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Operator Algebras-A Tool Kit

Exercise Sheet 11

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University of Helsinki, Fall 2014

Exercise 1: Show that if A is a unital C^*-algebra, then $U_n(A)$ is canonically isomorphic to $U(M_n(A))$.

Exercise 2: Show that $U_n(\mathbb{C})$ is connected for each $n \in \mathbb{N}$ and conclude that $K_1(\mathbb{C})$ is trivial. Proceed as follows:

(a) Let $u \in U_n(\mathbb{C})$. Argue that there exists an orthonormal basis $v_1, \ldots, v_n$ of eigenvectors of u.

(b) Let $\lambda_1, \ldots, \lambda_n$ denote the corresponding eigenvalues. Show that $|\lambda_j| = 1$ holds for each $1 \leq j \leq n$.

(c) Choose $\theta_j \in \mathbb{R}$ with $\lambda_j = e^{i\theta_j}$ and define a continuous curve by

$$\gamma : [0, 1] \to U_n(\mathbb{C}), \;\; \gamma(t).v_j := e^{ti\theta_j} \cdot v_j.$$

Show that $\gamma(0) = 1$ and $\gamma(1) = u$. Moreover, show that each $\gamma(t)$ is unitary, i.e., that the map γ is well-defined.

Exercise 3: Prove Proposition 6.2.4. Hint: Let $u, v \in U_n(A)$. Use 6.1.3 to show that $\operatorname{diag}(uv, 1)$, $\operatorname{diag}(vu, 1)$ and $\operatorname{diag}(u, v)$ are all in the same connected component of $U_{2n}(A)$.

Exercise 4: Show that if $\varphi : A \to B$ is a surjective *-homomorphism, then $\varphi(U(A)_0) = U(B)_0$. Proceed as follows:

(a) Let $A_{\text{sa}} := \{a \in A : \, a^* = a\}$. Show that $\varphi(A_{\text{sa}}) = B_{\text{sa}}$.

(b) Use the fact that every unitary in $U(A)_0$ (and also of $U(B)_0$) is a finite product of exponentials of the form e^{ix} for $x = x^*$ (cf. Exercise sheet 4, Exercise 1), to show that $\varphi(U(A)_0) = U(B)_0$.