8 September 2014

Operator Algebras-A Tool Kit

Exercise Sheet 1

Department of Mathematics and Statistics
University of Helsinki, Fall 2014

Exercise 1: Let A be a unital C*-algebra with unit given by 1_A. Show that the unit element is unique and satisfies $1_A^* = 1_A$ and $\|1_A\| = 1$.

Exercise 2: Show that each C*-algebra $(A, \|\cdot\|, *)$ is actually a topological algebra with respect to the topology induced by the norm $\|\cdot\|$, i.e., show that the vector-space operations and the multiplication map are continuous with respect to $\|\cdot\|$. Conclude that the maps

$$\lambda_a : A \to A,\ \ x \mapsto ax \ \text{ and } \ \rho_a : A \to A,\ \ x \mapsto xa$$

are continuous for all $a \in A$.

Exercise 3: Let $(A, \|\cdot\|_A, *)$ be a C*-algebra without unit and $\widetilde{A} := \mathbb{C} \times A$. For $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$ define the following operations on $\widetilde{A}$:

(1) $(\lambda, a) + (\mu, b) := (\lambda + \mu, a + b)$

(2) $\mu \cdot (\lambda, a) := (\mu\lambda, \mu a)$

(3) $(\lambda, a) \cdot (\mu, b) := (\lambda\mu, \lambda b + \mu a + ab)$

(4) $(\lambda, a)^* := (\overline{\lambda}, a^*)$

(5) $\|(\lambda, a)\|_{\widetilde{A}} := \sup_{\|b\|_A = 1} \|\lambda b + ab\|_A$.

Show that $(\widetilde{A}, \|\cdot\|_{\widetilde{A}}, *)$ is a unital C*-algebra and that A can be identified with the closed *-invariant subalgebra $\{0\} \times A \subseteq \widetilde{A}$.

Exercise 4: Show that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n$, i.e., that $\widehat{\mathbb{R}}^n \cong \mathbb{S}^n$. Hint: Use the stereographic projection.

Exercise 5: Let X be a locally compact Hausdorff space and $\widehat{X}$ be the corresponding one-point compactification of X. Show that the map

$$\Phi : C(\widehat{X}) \to \widetilde{C_0(X)},\ \ f \mapsto (f(\infty), (f - f(\infty))_{|X})$$

is an isomorphism of unital C*-algebras.

Exercise 6: Assume that you already know that each commutative C*-algebra is isomorphic to $C_0(X)$ for some locally compact space X. Justify the following statements:

(a) "*Unital C*-algebras can be thought of as algebraic generalizations* (*or so-called noncommutative versions*) *of compact Hausdorff spaces*".

(b) "*Unitalization of a C*-algebra can be thought of as the algebraic generalization* (*or so-called noncommutative version*) *of the topological concept of one-point compactification of a locally compact space*".