9.1 (In)Stability of Steady States

The presentation of this topic will be sketchy and sloppy. To begin with, we will <u>not</u> treat the initial value problem (see for instance nonlineardiffusioncwi syllabus.pdf, Chapter II and the references given there) and hence we do not properly define the dynamical system that is involved in the definition of stability. Secondly, our presentation of the maximum principle and the corresponding comparison principle will be rather superficial. Our aim is to illustrate ideas, methods and results by presenting them in the context of simple examples.

We start out by presenting the Principle of Linearized Stability in a general abstract context, as a sort of wishful thinking about a general theory for <u>systems</u> of nonlinear diffusion equations. Next we mainly focus on <u>scalar</u> equations and consider three key aspects

- the sign of the principal eigenvalue
- the Maximum Principle, the Comparison Principle, Sub- and Supersolutions
- Gradient systems and a Lyapunov functional

For equations with no-flux boundary conditions we ask the question:

Can a steady state that shows spatial structure, i.e., that is non-constant, be stable?

And similarly we ask for equations with big monster boundary conditions:

Can a steady state that changes sign be stable?

9.2 The Principle of Linearized Stability

Let $\Sigma : \mathbb{R}_+ \times X \to X$, where X is a Banach space, be such that

$$\Sigma(t+s, \phi) = \Sigma(t, \Sigma(s, \phi)) \qquad t, s \geq 0$$

We call Σ a <u>dynamical system</u>. When

$$\Sigma(t, \phi_e) = \phi_e \qquad \forall\, t \geq 0$$

we call ϕ_e a <u>steady state</u>. We call this steady state (locally) <u>stable</u> if

$\forall \varepsilon > 0 \ \exists\, \delta = \delta(\varepsilon) > 0$ such that

$$\|\phi - \phi_e\| \leq \delta \Rightarrow \|\Sigma(t, \phi) - \phi_e\| \leq \varepsilon$$

and <u>asymptotically stable</u> if, in addition, for some $\varepsilon_0 > 0$

$$\lim_{t \to \infty} \|\Sigma(t, \phi) - \phi_e\| = 0$$

for all ϕ with $\|\phi - \phi_e\| \leq \varepsilon_0$. When ϕ_e is not stable we say that ϕ_e is <u>unstable</u>. When for some $\varepsilon_0 > 0$, ~~Mzt~~ $K > 0$ and $\omega < 0$ the estimate

$$\|\Sigma(t, \phi) - \phi_e\| \leq K e^{\omega t}$$

holds for $t \geq 0$ and ϕ with $\|\phi - \phi_e\| \leq \varepsilon_0$, we say that ϕ_e is (locally) <u>exponentially stable</u>.

Let for each $t \geq 0$ the operator

$$T(t) : X \to X$$

be linear and bounded and assume that

9.3/ i) $T(0) = I$ ii) $T(t+s) = T(t)T(s)$, $t,s \geq 0$

(in words: $\{T(t)\}$ is a semigroup of bounded linear operators)

Assume that these $T(t)$ are the linearization of

$$\phi \longmapsto \left(\Sigma(t, \phi_e + \phi) - \phi_e \right)$$

at zero in the following sense

$\forall T>0, \varepsilon>0 \ \exists \ \delta = \delta(T,\varepsilon) > 0$ such that

$$\|\phi\| \leq \delta \quad \Rightarrow \quad \underset{0 \leq t \leq T}{\| \Sigma(t, \phi_e + \phi) - \phi_e - T(t)\phi \|} \leq \varepsilon \|\phi\|$$

Theorem 1 Assume that $\gamma > 0$ and $M \geq 1$ exist such that for $t \geq 0$

$$\|T(t)\| \leq M e^{-\gamma t}$$

Then ϕ_e is (locally) exponentially stable as a steady state of Σ

Proof Assume, without loss of generality, that $\phi_e = 0$. Choose t_0 such that $M e^{-\gamma t_0} \leq \frac{1}{4}$. Choose $\delta > 0$ such that for $\|\phi\| \leq \delta$ and $0 \leq t \leq t_0$

$$\| \Sigma(t, \phi) - T(t)\phi \| \leq \frac{1}{4} \|\phi\|$$

Then for such ϕ and t we have

$$\begin{aligned} \| \Sigma(t,\phi) \| &\leq \| \Sigma(t,\phi) - T(t)\phi \| + \| T(t)\phi \| \\ &\leq \frac{1}{4} \|\phi\| + M e^{-\gamma t} \|\phi\| \\ &\leq \left(\frac{1}{4} + M \right) \|\phi\| \end{aligned}$$

while for $t = t_0$ we also have the sharper estimate

$$\| \Sigma(t_0, \phi) \| \leq \frac{1}{2} \|\phi\|$$

9.4/ So in particular $\|\Sigma(t_0,\phi)\| < \delta$ and we may therefore iterate to obtain

$$\|\Sigma(kt_0,\phi)\| \leq \left(\tfrac{1}{2}\right)^k \|\phi\|$$

For $t>0$ put $\quad t = k t_0 + \tau \quad$ with $0 \leq \tau < t_0$

For $\|\phi\| \leq \delta\left(\frac{1}{2}+M\right)^{-1}$ we have

$$\|\Sigma(\tau,\phi)\| \leq \left(\tfrac{1}{4}+M\right)\|\phi\| \leq \delta$$

Hence

$$\|\Sigma(t,\phi)\| = \|\Sigma(kt_0,\Sigma(\tau,\phi))\| \leq \left(\tfrac{1}{2}\right)^k \|\Sigma(\tau,\phi)\|$$

$$\leq \left(\tfrac{1}{2}\right)^k\left(\tfrac{1}{4}+M\right)\|\phi\| \leq \left(\tfrac{1}{4}+M\right)e^{-\frac{\log 2}{t_0}t}\|\phi\| \quad \square$$

Theorem 1 shows that an exponential estimate for the linearized dynamical system can be "lifted" to an exponential estimate for the nonlinear dynamical system. Our next task is to show (that) exponential growth of certain solutions of the linear system implies instability of the steady state of the nonlinear system. Again our strategy will be to first look at iteration of a discrete time map.

Proposition Let ϕ_e be a fixed point of a map $F: X \to X$. Assume that F is differentiable at ϕ_e and denote $DF(\phi_e)$ by L. Assume that X admits a decomposition

$$X = X_- \oplus X_+$$

with both X_- and X_+ invariant under L and closed, 9.5/ and such that for some $\theta \geq 1$ and $\delta > 0$

i) $\|L\phi\| \leq \theta \|\phi\|$ for $\phi \in X_-$

ii) $\|L\phi\| \geq (\theta+\delta)\|\phi\|$ for $\phi \in X_+$

Then ϕ_e is unstable as a fixed point of the iteration scheme

$$x(n+1) = F(x(n))$$

Proof Again assume $\phi_e = 0$. Let P denote the projection on X_+ along X_-. Define

$$|||\phi||| = \|P\phi\| + \|(I-P)\phi\|$$

then $|||\cdot|||$ is an equivalent norm on X. Choose $\varepsilon > 0$ such that for $|||\phi||| < \varepsilon$ we have

$$|||F(\phi) - L\phi||| \leq \tfrac{1}{4}\delta\, |||\phi||| \qquad \circledast$$

For ϕ such that $|||\phi||| < \varepsilon$ and $\|(I-P)\phi\| \leq \|P\phi\|$ we then have

$$\|PF(\phi)\| \geq \|PL\phi\| - \|P(F(\phi)-L\phi)\|$$

$$\geq (\theta+\delta)\|P\phi\| - \tfrac{1}{4}\delta\, |||\phi||| \quad \leftarrow \text{since } |||\phi||| \leq 2\|P\phi\|$$

$$\geq (\theta+\delta)\|P\phi\| - \tfrac{1}{2}\delta\|P\phi\|$$

$$\Rightarrow \|PF(\phi)\| \geq \left(\theta + \frac{\delta}{2}\right)\|P\phi\|$$

Likewise

$$\|(I-P)F(\phi)\| \leq \|(I-P)L\phi\| + \|(I-P)(F(\phi)-L\phi)\|$$

$$\leq \theta\|(I-P)\phi\| + \tfrac{1}{2}\delta\|P\phi\|$$

$$\leq \left(\theta + \frac{\delta}{2}\right)\|P\phi\|$$

since $\circledast$ implies $\|(I-P)(F(\phi)-L\phi)\| \leq \frac{1}{4}\delta\left(\|P\phi\| + \|(I-P)\phi\|\right)$

Hence $F(\phi)$ satisfies the cone condition

$$\|(I-P)F(\phi)\| \leq \|PF(\phi)\|$$

Now suppose that $\|F^{(n)}(\phi)\| \leq \varepsilon$ for all n, for ϕ with $\|\phi\|$ sufficiently small. Then

$$\|PF^{(n)}(\phi)\| \geqslant \left(\theta + \frac{\delta}{2}\right)^n \|P\phi\|$$

which, since $\theta + \frac{\delta}{2} > 1$, tends to ∞ as $n \to \infty$ whenever $\|P\phi\| > 0$. So the assumption that $\|F^{(n)}(\phi)\| \leq \varepsilon$ for all n cannot hold. We conclude that nonzero initial points in the cone segment

$$\{\phi : \|P\phi\| \geqslant \|(I-P)\phi\| \ \& \ |||\phi||| \leq \varepsilon\}$$

lead to orbits which necessarily must leave the ε-ball $\square$

<u>Theorem 2</u> Assume that X admits a decomposition

$$X = X_- \oplus X_+$$

into $T(t)$-invariant closed subspaces, with X_+ being finite dimensional. Let $A_+ : X_+ \to X_+$ be the bounded linear operator such that

$$T(t)\Big|_{X_+} = e^{tA_+}$$

Assume that $M \geqslant 1$ and $\omega \geqslant 0$ exist such that

(i) $\quad \|T(t)|_{X_-}\| \leq Me^{\omega t}$

(ii) $\quad \operatorname{Re}\lambda > \omega$ for all $\lambda \in \sigma(A_+)$

9.7/ Then ϕ_e is unstable as a steady state of Σ.

Proof By (ii) we know that $\varepsilon > 0$ exists such that $\operatorname{Re}\lambda > \omega + \varepsilon$ for all $\lambda \in \sigma(A_+)$. We translate this into

$$\operatorname{Re}\lambda < -(\omega+\varepsilon) \quad \text{for all} \quad \lambda \in \sigma(-A_+)$$

It follows that for every $\tilde{x}_+ \in X_+$ the estimate

$$\| e^{-A_+ t} \tilde{x}_+ \| \leq K e^{-(\omega+\varepsilon)t} \| \tilde{x}_+ \|$$

holds for some $K \geq 1$. Now put $\tilde{x}_+ = e^{tA_+} x_+$ then

$$\| x_+ \| \leq K e^{-(\omega+\varepsilon)t} \| e^{tA_+} x_+ \|$$

and hence
$$\| e^{tA_+} x_+ \| \geq \frac{1}{K} e^{(\omega+\varepsilon)t} \| x_+ \|$$

Choose $s \in \mathbb{R}_+$ so large that $\frac{1}{K} e^{(\omega+\varepsilon)s} > M e^{\omega s}$.
Apply the proposition with:

$$L = T(s), \quad F(\phi) = \Sigma(s,\phi), \quad \theta = M e^{\omega s},$$

$\delta = \frac{1}{K} e^{(\omega+\varepsilon)s} - M e^{\omega s}$. Finally, note that for this instability result the discrete time assertion suffices, i.e., there is no need to consider the times in between two multiples of s. $\square$

<u>Theorem 3</u> (Principle of Linearized Stability)
Let ϕ_e be a steady state of the dynamical system Σ and assume that, uniformly on compact sets $0 \leq t \leq t_0$, the map $\phi \mapsto \Sigma(t,\phi)$ has derivative $\phi \mapsto T(t)\phi$ at ϕ_e. Assume that X admits a decomposition

$$X = X_- \oplus X_+$$

into $T(t)$-invariant closed subspaces, with X_+ being finite dimensional. Assume that the restriction of $T(t)$ to X_- converges to zero exponentially as $t \to \infty$. Let the restriction of $T(t)$ to X_+ be given by e^{tA_+}. Then ϕ_e is

i) (locally) exponentially stable when $\operatorname{Re}\lambda < 0 \quad \forall \lambda \in \sigma(A_+)$

ii) unstable when there exists $\lambda \in \sigma(A_+)$ with $\operatorname{Re}\lambda > 0$

PS The text for this subsection was taken from O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, H.-O. Walther Delay Equations: functional-, complex- and nonlinear analysis, Springer, 1995

In order to analyse nonlinear diffusion problems along the lines of the exposition above, one needs

① to show that the solution operators of the nonlinear problem are indeed differentiable and that their derivatives are the solution operators of the corresponding linearized problem

② to derive exponential estimates for the linear problem, possibly after introducing a decomposition of the state space X

Point ① involves, in one way or another, the variation-of-constants formula, also known as Duhamel's Formula.

9.9/ Point ② is straightforward in the L_2-setting but, as explained before, L_2 is not a good space for nonlinear problems. As a conseauence, also point ② requires technical work that is far from trivial.

Anyhow, I know far too little about this subject to lecture about it. For inspiration I refer to

D. Henry, Geometric theory of semilinear equations
Springer LNiM 840, 1981

A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Birkhäuser, 1995

Parabolic problems arising in population biology

Let's focus on

$$d = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_k \end{pmatrix}$$

$$\frac{\partial u}{\partial t} = d \Delta u + f(u) \qquad x \in \Omega \subset \mathbb{R}^n$$

with boundary conditions

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0 \text{ (no flux)} \quad \text{or} \quad u\Big|_{\partial\Omega} = 0 \text{ (big monster)}$$

and ~~we~~ where $u(t,x)$ might be a vector (say $u(t,x) \in \mathbb{R}^k$ and $f: \mathbb{R}^k \to \mathbb{R}^k$; carefully note that the dimension of the spatial variable x is denoted by n while the dimension of the system of equations is denoted by k). Suppose that we know (for instance via bifurcation analysis) a steady state ϕ_e, i.e., a solution of

$$0 = d\Delta \phi_e + f(\phi_e)$$

$$\frac{\partial \phi_e}{\partial n}\Big|_{\partial\Omega} = 0 \qquad \text{or} \qquad \phi_e\Big|_{\partial\Omega} = 0$$

9.10 The linearized problem is then given by

$$\frac{\partial w}{\partial t} = d\Delta w + Df(\phi_e)w$$

$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0 \qquad \text{or} \qquad w\big|_{\partial\Omega} = 0$$

and leads to the eigenvalue problem

$$d\Delta\psi + Df(\phi_e)\psi = \lambda\psi$$

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = 0 \qquad \text{or} \qquad \psi\big|_{\partial\Omega} = 0$$

For scalar equations, i.e., $k=1$, we know that there exists a principal eigenvalue λ_d (d for dominant) with corresponding positive eigenfunction ψ_d. Recall that λ_d is real. Motivated by our earlier discussion we now simply call

ϕ_e stable if $\lambda_d < 0$

ϕ_e unstable if $\lambda_d > 0$

even though we haven't yet justified this terminology (below we shall do so, using maximum principle arguments and a Lyapunov function). For systems of equations we shall also call ϕ_e stable if $\mathrm{Re}\,\lambda < 0$ for all eigenvalues λ and call ϕ_e unstable if $\mathrm{Re}\,\lambda > 0$ for some eigenvalue λ, but for systems we give no justification at all.

No-flux boundary conditions

In case of no-flux boundary conditions, any solution of the ODE

$$\frac{d\tilde{u}}{dt} = f(\tilde{u})$$

yields a uniform (i.e., "flat") solution of the PDE via

$$u(t,x) = \tilde{u}(t) \quad , \; x \in \Omega$$

Q: An obvious question is: does stability wrt the ODE guarantee stability wrt the PDE? For a scalar equation ($k=1$) the eigenvalues are obtained by shifting the eigenvalues λ_j of the Laplace operator (*) with no-flux boundary conditions over $f'(\bar{u})$, where $\bar{u}$ is such that $f(\bar{u})=0$, so $\bar{u}$ is the uniform/flat steady state that we consider. Since $\lambda_j \leq 0$, clearly $f'(\bar{u}) < 0$ implies that all eigenvalues are negative. So for $k=1$ the

A: answer to the above question is: yes.

(*) Take $d=1$ without loss of generality.

Now consider $k>1$ and in fact $k=2$, for ease of formulation. In a later lecture on diffusion driven instability (also known as Turing instability) we shall show that for the system

$$\frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + f_1(u_1,u_2)$$

$$\frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + f_2(u_1,u_2)$$

the matrix

$$Df(\bar{u}) + \lambda_j d$$

may have an eigenvalue with positive real part even if $Df(\bar{u})$ itself has only eigenvalues with negative real part. So for systems of equations, i.e., $k \geq 2$, the answer to the above question is: no, not necessarily.

A:

Using bifurcation analysis of the Turing instability, one can show that for systems of equations with no-flux boundary conditions, non-flat steady states can be stable. The question arises:

Q: Can a non-constant (in space) solution of a scalar equation be stable?

A: In one space dimension ($n=1$) the answer is: no

Proof Let ϕ_e satisfy

$$\phi_e'' + f(\phi_e) = 0$$
$$\phi_e'(0) = 0 = \phi_e'(1)$$

Define $\psi = \phi_e'$ then, by differentiation of the equation we find that ψ satisfies

$$\psi'' + f'(\phi_e)\psi = 0$$
$$\psi(0) = 0 = \psi(1)$$

while the boundary conditions follow directly from $\psi = \phi_e'$ and the boundary conditions for ϕ_e. We can translate these identities into the statement: the linearized problem with big ~~moste~~ monster boundary conditions has an eigenvalue zero. So the principal eigenvalue with these boundary conditions is bigger than or equal to zero. From the variational characterization of the principal eigenvalue in terms of a Rayleigh quotinent, we know that

9.13 ~~the principal eigenvalue with no-flux boundary conditions is strictly greater ~~that~~ than zero.~~ $\square$

A: In any space dimension, if Ω is convex, the answer is : no

<u>Proof</u> We now differentiate the equation for ϕ_e with respect to x_j

$$\Rightarrow \quad \Delta\left(\frac{\partial \phi_e}{\partial x_j}\right) + f'(\phi_e)\frac{\partial \phi_e}{\partial x_j} = 0$$

Next multiply this equation by $\frac{\partial \phi_e}{\partial x_j}$ and integrate-by-parts

$$0 = \int_\Omega \left(-\left|\nabla \frac{\partial \phi_e}{\partial x_j}\right|^2 + f'(\phi_e)\left(\frac{\partial \phi_e}{\partial x_j}\right)^2 \right) + \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial n}\left(\frac{\partial \phi_e}{\partial x_j}\right)^2$$

<u>Exercise</u> verify the last step

Below we shall prove

<u>Lemma</u> Let Ω be convex and let $w \in C^2(\bar{\Omega})$ with $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$. Then $\frac{\partial}{\partial n}|\nabla w|^2 \leq 0$ on $\partial\Omega$

If we sum j over $1, 2, \dots, n$ and use the lemma we obtain

$$\sum_{j=1}^{n} \int_\Omega \left(\left|\nabla \frac{\partial \phi_e}{\partial x_j}\right|^2 - f'(\phi_e)\left(\frac{\partial \phi_e}{\partial x_j}\right)^2 \right) \leq 0$$

If the sum is negative, at least one of the terms is negative. So then the variational characterization of the principal eigenvalue in terms of the Rayleigh quotient on H^1 implies that $\lambda_d > 0$. It remains to deal with the case that all terms are equal to zero. Note that, since by assumption ϕ_e is not constant, we have (that) $\partial\phi_e/\partial x_j$ differs from the null function for at least one index j. So if $\lambda_d \leq 0$ it is in fact equal to zero, again on account of the variational characterization.

9.14/ Let ψ_d denote the corresponding eigenfunction. Recall that λ_d is simple.

From $\int_\Omega \left(\left| \nabla \frac{\partial \phi_e}{\partial x_j} \right|^2 - f'(\phi_e) \left(\frac{\partial \phi_e}{\partial x_j} \right)^2 \right) = 0$ we conclude that

$\partial \phi_e / \partial x_j = c_j \psi_d$ (with possibly $c_j = 0$ but $c_i \neq 0$ for at least one index i since ϕ_e is not constant). Let

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

then ϕ_e is constant along lines orthogonal to c and therefore

$$\phi_e(x_1, \dots, x_n) = \Phi(c_1 x_1 + \dots + c_n x_n)$$

for some function Φ of one variable. It follows that

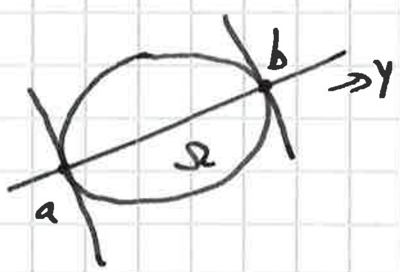
$$\Delta \phi_e(x) = \sum_{i=1}^n c_i^2 \, \Phi''(y) \qquad \text{with } y = c_1 x_1 + \dots + c_n x_n$$

and hence

$$\sum c_i^2 \, \Phi'' + f(\Phi) = 0$$

Choose the hyperplanes $y = a$ and $y = b$ such that for $x \in \Omega$ we have $a \leq y \leq b$ and $y = a$ as well as $y = b$ correspond to points of $\partial\Omega$. Next note that for $x \in \partial\Omega$ corresponding to $a = y$

$$0 = \frac{\partial \phi_e}{\partial n} = \sum_{i=1}^n \nu_i \frac{\partial \phi_e}{\partial x_i} = \sum_{i=1}^n \nu_i c_i \, \Phi'(a)$$



It is impossible that $\sum_{i=1}^n \nu_i c_i = 0$ since that would imply that the vector c is tangent to $\partial\Omega$ in x. Hence $\Phi'(a) = 0$.

Suppose $\Phi''(a) = 0$ as well. Then the equation for Φ implies that $f(\Phi(a)) = 0$. But then $\Phi(y) = \Phi(a)$

 is a solution of the equation for Φ and by the uniqueness theorem for solutions of systems of ODE there is no other solution with the same boundary value $\Phi(a)$ and satisfying $\Phi'(a)=0$. So then Φ is constant and, after all, ϕ_e is constant, a contradiction. So then we find that λ_d cannot be zero and must be positive.

We therefore set out to show that $\Phi''(a)=0$. First note that from $\frac{\partial \phi_e}{\partial x_j} = c_j \psi_d$ it follows that $\frac{\partial}{\partial n}\left(\frac{\partial \phi_e}{\partial x_j}\right) = c_j\left(\frac{\partial \psi_d}{\partial x_j}\right) = 0$ at $\partial\Omega$, since ψ_d is eigenfunction for the no-flux problem. So

$$0 = \frac{\partial}{\partial n}\left(\frac{\partial \phi_e}{\partial x_j}\right) = \sum_{i=1}^{n} \nu_i \frac{\partial^2 \phi_e}{\partial x_i \partial x_j} = \sum_{i=1}^{n} \nu_i c_i c_j \Phi''(a)$$

and we conclude that $c_j \Phi''(a) = 0$ for $j=1,\dots,n$. Since at least one of the c_j is non-zero, we must have $\Phi''(a)=0$ $\square$

<u>Proof of the Lemma</u> stated on page 9.13. Let $P \in \partial\Omega$. After rotation and translation, we may assume that P is located at the origin and that the boundary $\partial\Omega$ near P is represented by $x_n = h(\tilde{x})$ where $\tilde{x} = (x_1,\dots,x_{n-1})$ with

$$\tilde{\nabla} h(0) = \left(\frac{\partial h}{\partial x_1}(0),\dots,\frac{\partial h}{\partial x_{n-1}}(0)\right)^T = 0$$

and the outer normal unit vector is given by $\nu = (0,\dots,0,-1)$. By the convexity of Ω at 0, the Hessian $\frac{\partial^2 h}{\partial x_i \partial x_j}(0)$ is nonnegative definite. For $x \in \partial\Omega$ near P we have

$$\nu = \frac{(\tilde{\nabla} h(x), -1)^T}{|(\tilde{\nabla} h(x), -1)|}$$

Hence

$$0 = \frac{\partial w}{\partial n} = \nu \cdot \nabla w = \frac{\sum_{i=1}^{n-1} \frac{\partial h}{\partial x_i}(x)\frac{\partial w}{\partial x_i}(x) - \frac{\partial w}{\partial x_n}(x)}{\left(\sum_{j=1}^{n-1}\left(\frac{\partial h}{\partial x_j}(x)\right)^2 + 1\right)^{1/2}}$$

9.16/ which implies

$$(*) \qquad \frac{\partial w}{\partial x_n}(x) = \sum_{i=1}^{n-1} \frac{\partial w}{\partial x_i}(x)\frac{\partial h}{\partial x_i}(x) \quad \text{for } x\in\partial\Omega,$$

x small, so for $(\tilde{x}, h(\tilde{x}))$ with $\tilde{x}$ small

Differentiating $(*)$ wrt x_j, $1\le j\le n-1$, we obtain

$$\frac{\partial^2 w}{\partial x_n \partial x_j} + \frac{\partial^2 w}{\partial x_n^2}\frac{\partial h}{\partial x_j} = \sum_{i=1}^{n-1}\left(\frac{\partial^2 w}{\partial x_i\partial x_j}\frac{\partial h}{\partial x_i} + \frac{\partial^2 w}{\partial x_i \partial x_n}\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_i} + \frac{\partial w}{\partial x_i}\frac{\partial^2 h}{\partial x_i \partial x_j}\right)$$

and in particular

$$(**) \qquad \frac{\partial^2 w}{\partial x_n \partial x_j}(0) = \sum_{i=1}^{n-1}\frac{\partial w}{\partial x_i}(0)\frac{\partial^2 h}{\partial x_i\partial x_j}(0) \qquad \text{since } \frac{\partial h}{\partial x_i}(0)=0,\ i=1,\dots,n-1$$

Now we calculate

$$\frac{\partial}{\partial n}|\nabla w|^2(0) = -\frac{\partial}{\partial x_n}|\nabla w|^2(0) = -2\left\{\sum_{j=1}^{n-1}\left(\frac{\partial w}{\partial x_j}\frac{\partial^2 w}{\partial x_j\partial x_n} + \frac{\partial w}{\partial x_n}\frac{\partial^2 w}{\partial x_n^2}\right)\right\}(0)$$

$$\overset{\substack{\text{by } (**) \\ \text{and } \frac{\partial w}{\partial x_n}(0) = -\frac{\partial w}{\partial n}(0)=0}}{=} -2\sum_{i,j=1}^{n-1}\frac{\partial w}{\partial x_i}(0)\frac{\partial w}{\partial x_j}(0)\frac{\partial^2 h}{\partial x_i\partial x_j}(0) \le 0$$

Hessian $\frac{\partial^2 h}{\partial x_i \partial x_j}$ is nonnegative definite

$\square$

A: If either $f'' > 0$ or $f'' < 0$ on a closed interval that contains the range of ϕ_e, the answer is: no

<u>Proof</u> for the case $f''>0$. We ~~recall~~ recall that

$$\lambda_d = \sup_{\psi\in H^1(\Omega)} \frac{\int_\Omega \left(-|\nabla\psi|^2 + f'(\phi_e)\psi^2\right)}{\int_\Omega \psi^2}$$

Let

$$c = \min_{x\in\bar{\Omega}} \phi_e(x)$$

9.17 We shall show that for $\psi = \phi_e - c$ the Rayleigh quotient assumes a positive value. In preparation we first observe that $\Delta \phi_e + f(\phi_e) = 0$, $\frac{\partial \phi_e}{\partial n} = 0$ implies

i) $-\int_\Omega |\nabla \phi_e|^2 + \int_\Omega \phi_e f(\phi_e) = 0$ (multiply the equation by ϕ_e and integrate by parts)

ii) $\int_\Omega f(\phi_e) = -\int_\Omega \Delta \phi_e = \int_{\partial\Omega} \frac{\partial \phi_e}{\partial n} = 0$

For $\psi = \phi_e - c$ the numerator of the Rayleigh quotient is

$$\int_\Omega \left(-|\nabla \phi_e|^2 + f'(\phi_e)(\phi_e - c)^2\right) \underset{\substack{\uparrow \\ \text{by i) and ii)}}}{=} \int_\Omega \left((-\phi_e + c) f(\phi_e) + f'(\phi_e)(\phi_e - c)^2\right)$$

$$= -\int_\Omega (\phi_e - c)\{f(\phi_e) - f'(\phi_e)(\phi_e - c)\} = (*)$$

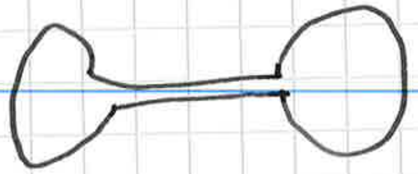
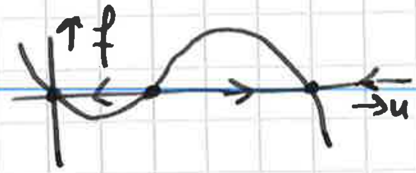
Clearly $\Delta \phi_e \geq 0$ in points in the interior of Ω where ϕ_e assumes its minimal value c, so if there are such points we conclude that necessarily $f(c) \leq 0$. But even if c is only assumed at $\partial\Omega$, $f(c) > 0$ would imply that $\Delta \phi_e < 0$ in a neighbourhood and, since $\frac{\partial \phi_e}{\partial n} = 0$ at $\partial\Omega$, ϕ_e would assume values less than c, a contradiction. So $f(c) \leq 0$. Since $f'' > 0$

$$f(c) > f(\phi_e(x)) + f'(\phi_e(x))(c - \phi_e(x)) \quad \text{if } \phi_e(x) \neq c$$

so

$$f(\phi_e(x)) - f'(\phi_e(x))(\phi_e(x) - c) < f(c) \leq 0 \quad \text{if } \phi_e(x) > c$$

We conclude that $(*) > 0$ $\square$

9.18/ For Ω =  and 

A: bistable f the answer is: YES

see H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. RIMS, Kyoto Univ. 15 (1979) 401–451

which contains much more material than explained above.

Other relevant references are:

R. G. Casten, C. J. Holland, Instability results for reaction diffusion equations with Neumann boundary conditions, J. Diff. Equ. 27 (1978) 266–273

N. Chafee, Asymptotic behavior for solutions of a one-dimensional parabolic equation with homogeneous Neumann boundary conditions, J. Diff. Equ. 18 (1975) 111–134

K. Kishimoto, H. F. Weinberger, The spatial homogeneity of stable equilibria of some reaction – diffusion systems on convex domains, J. Diff. Equ. 58 (1985) 15–21

Wei-Ming Ni, The Mathematics of Diffusion, CBMS-NSF Regional Conference Series in Appl. Math. 82, SIAM, 2011

Conclusion For scalar equations with the "neutral" no-flux boundary conditions, diffusion has a flattening effect, i.e., eliminates any spatial heterogeneity that might initially exist. (Strictly speaking this is not true: if there are multiple stable steady states for the ODE and the domain consists of subdomains that are connected by narrow corridors, a stable solution exists that is close to the value of different stable ODE steady states in different subdomains.) In the forthcoming section on Turing instability we will see that for systems of equations, the kinetic ODE part and the diffusion part may generate patterns

Big Monster Boundary Conditions

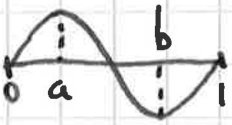
If the ODE kinetics are such that solutions grow away from zero, but, by imposing boundary conditions, we force the solution to be zero at the boundary, the balance between the two opposing forces automatically generates a pattern. But if the coefficients of the equation do not depend on the spatial variable x, we expect a relatively simple pattern: positive in the interior, zero at the boundary. By way of phase plane analysis we have found that there are in fact steady states that do show changes of sign. (Indeed, we even classified the solutions by the number of sign changes.)

Q: Can a solution that changes sign be stable?

A: If $k=1$ (scalar equation) and $n=1$ (one dimensional space) the answer is: no

$f(0)=0$

<u>Proof</u> Assume ϕ_e satisfies $\phi_e'' + f(\phi_e)=0$, $\phi_e(0)=0=\phi_e(1)$ and ϕ_e assumes both positive and negative values on $[0,1]$



Then ϕ_e' has at least two zero's in $(0,1)$, say a,b with $a<b$. So if we put $\psi = \phi_e'|_{[a,b]}$ then

$$\psi'' + f'(\phi_e)\psi = 0 \quad , \quad \psi(a) = 0 = \psi(b)$$

or, in words, $\lambda=0$ is an eigenvalue of the "big monster" problem on $[a,b]$ with corresponding eigenfunction ψ. If ψ itself changes sign on $[a,b]$ we infer that the principal eigenvalue on $[a,b]$ is strictly positive and hence so is the principal eigenvalue on $[0,1]$. So assume that ψ itself is positive on (a,b) (if it is actually negative, consider $-\psi$). Then necessarily

$$\psi'(a) > 0 \quad \text{and} \quad \psi'(b) < 0$$

since if either $\psi'(a)=0$ or $\psi'(b)=0$ we find that ψ is identically zero by uniqueness of solutions of the two dimensional system of ODE corresponding to the second order equation for ψ. Let λ_d be the ~~der~~ principal eigenvalue on $[0,1]$ and ψ_d the correrponding eigenfunction, then $\psi_d(a)>0$ and $\psi_d(b)>0$. So on $[a,b]$ we have

$$\psi_d'' + f'(\phi_e)\psi_d = \lambda_d \psi_d \quad , \quad \psi_d(a)>0,\ \psi_d(b)>0$$

$$\psi'' + f'(\psi_e)\psi = 0 \quad , \quad \psi(a) = 0 = \psi(b)$$

Multiply the first equation by ψ, the second by ψ_d, integrate both over $[a,b]$ and subtract the second from the first. This yields the identity

$$\int_a^b \psi_d''\psi - \int_a^b \psi''\psi_d = \lambda_d \int_a^b \psi_d \psi$$

Now integrate both terms at the left hand side by parts to obtain

$$\left[\psi_d'\psi - \psi'\psi_d\right]_a^b = -\psi'(b)\psi_d(b) + \psi'(a)\psi_d(a) > 0$$

Since also $\int_a^b \psi_d\psi > 0$ the conclusion is that $\lambda_d > 0$ $\square$

For extensions of this result to higher space dimension, as well as for counterexamples, see

E.N. Dancer, Z.M. Guo, Some remarks on the stability of sign changing solutions, Tôhoku Math. J. **47** (1995) 199-225

and the references given there.

The Maximum Principle

M. H. Protter & H. F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, 1967

The present section serves as a brief introduction to a very important tool for the study of scalar reaction diffusion equations : the Maximum Principle

Let u be C^2 on $(0,1)$ and assume u has a local maximum in $y \in (0,1)$ then necessarily

$$u'(y) = 0, \quad u''(y) \leq 0$$

so if we know that

$$u''(x) + g(x)u'(x) > 0 \quad \text{for } x \in (0,1)$$

for a bounded function g, we arrive at a contradiction and no such y can exist. So if we also know that $u \in C[0,1]$ we conclude that

$$u(x) \leq \max\{u(0), u(1)\}$$

with strict inequality for $x \in (0,1)$

But what can we say if we only know that

$$u''(x) + g(x)u'(x) \geq 0 \quad \text{for } x \in (0,1) \quad ?$$

Of course u can now be constant on $[0,1]$, but let's assume that is not the case, then once more the inequality

$$u(x) < \max\{u(0), u(1)\} \quad , x \in (0,1)$$

holds. We use a trick to show this. Assume there exists $x_0 \in (0,1)$ such that $u(x_0) \geq \max\{u(0), u(1)\}$. Since u is not constant, there must exist $x_1 \in (0,1)$ such that $u(x_1) \neq u(x_0)$. Consider the case $x_1 > x_0$. Let

$$v(x) = e^{\alpha(x - x_0)} - 1$$

then

$$v''(x) + g(x)v'(x) = \alpha(\alpha + g(x))e^{\alpha(x-x_0)} > 0$$

for α sufficiently large. So for any $\varepsilon > 0$ we have for

$$w(x) = u(x) + \varepsilon v(x)$$

9.22 that

$$w''(x) + g(x)w'(x) > 0 \quad \text{on } (0,1)$$

and therefore w has no local maxima in $(0,1)$. Choose

$$\varepsilon < \frac{|u(x_0) - u(x_1)|}{v(1)}$$

Note that

i) $w(0) < u(x_0)$ (since $v(0)<0 \Rightarrow w(0) < u(0) \leq u(x_0)$)

ii) $w(x_0) = u(x_0)$

We claim that if $u(x_1) < u(x_0)$

iii) $w(x_1) < u(x_0)$

implying that w has a local maximum in $(0,x_1)$, a contradiction.

Proof of claim: $w(x_1) = u(x_1) + \varepsilon v(x_1) < u(x_1) + \frac{v(x_1)}{v(1)}(u(x_0) - u(x_1))$

$$= \frac{v(x_1)}{v(1)}u(x_0) + \left(1 - \frac{v(x_1)}{v(1)}\right)u(x_1) < \frac{v(x_1)}{v(1)}u(x_0) + \left(1 - \frac{v(x_1)}{v(1)}\right)u(x_0) = u(x_0) \quad \square$$

We know that $u(x_1) \neq u(x_0)$ so it remains to consider the case that $u(x_1) > u(x_0)$. We claim that in that case

iii)' $\quad w(1) < w(x_1)$

implying that w has a local maximum in $(x_0,1)$ (since now $w(x_1) > u(x_1) > u(x_0) = w(x_0)$), again a contradiction.

Proof of claim: $w(1) = u(1) + \varepsilon v(1) < u(1) + u(x_1) - u(x_0)$

$\leq u(x_1) < w(x_1) \quad \square$

If $x_1 < x_0$ consider $v(x) = e^{-\alpha(x-x_0)} - 1$ and follow the same line of argumentation.

Thus we proved:

Theorem 1 Assume u is twice continuously differentiable and satisfies the differential inequality

$$u''(x) + g(x)u'(x) \geq 0 \qquad , 0<x<1$$

where g is a bounded function. Then either

$$u(x) < \max\{u(0), u(1)\} \qquad , 0<x<1$$

or u is constant on $[0,1]$. Likewise

$$u''(x) + g(x)u'(x) \leq 0 \qquad 0<x<1$$

implies that either

$$u(x) > \min\{u(0), u(1)\} \qquad , 0<x<1$$

or u is constant on $[0,1]$.

And by similar arguments one can prove

Theorem 2 Assume u is twice continuously differentiable and satisfies the differential inequality

$$u''(x) + g(x)u'(x) + h(x)u(x) \geq 0 \qquad , 0<x<1$$

where both g and h are bounded and, moreover,

$$h(x) \leq 0 \qquad , 0<x<1$$

then u has no interior nonnegative maximum unless u is in fact constant. Likewise

$$u''(x) + g(x)u'(x) + h(x)u(x) \leq 0 \qquad , 0<x<1$$

where now

$$h(x) \geq 0 \qquad , 0<x<1$$

implies that u cannot have an interior nonpositive minimum unless u is constant.

Both theorems have an analogue in higher space dimension.

Along similar lines one can also prove a parabolic maximum principle:

Theorem 3 Let $D = \{(t,x) : a < x < b, 0 < t < T\}$ and let u be defined on $\bar{D}$ and such that u_x, u_{xx}, u_t are well-defined and continuous on $\tilde{D}$. Assume that

$$\frac{\partial^2 u}{\partial x^2} + g\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} \geqslant 0 \quad \text{in } \tilde{D}, \quad g \text{ bounded on } \bar{D}$$

$$\tilde{D} = D \cup \{(T,x) : a < x < b\}$$

Then

$$u(t,x) \leq \sup\{u(\tau,\xi) : (\tau,\xi) \in \partial D \text{ but } \tau < T\}$$

Theorem 4 In the same setting as the preceding theorem, assume that

$$\frac{\partial^2 u}{\partial x^2} + g\frac{\partial u}{\partial x} + hu - \frac{\partial u}{\partial t} \geqslant 0 \quad \text{in } D$$

where also h is bounded on $\bar{D}$ and, moreover, $h \leq 0$.

Then

$$u(t,x) \leq \max\{0, \sup\{u(\tau,\xi) : (\tau,\xi) \in \partial D \text{ but } \tau < T\}\}$$

By the trick of writing

$$f(w) - f(u) = \frac{f(w) - f(u)}{w - u}(w - u) \qquad w \neq u$$

one can use Theorem 4 to prove the following very useful Comparison Theorem

we shall use this for proving results about the asymptotic speed of propagation; therefore we take c a constant and allow a and b to be infinite

Theorem 5 Let $D = (a,b) \times (0,T)$, where $-\infty \leq a < b \leq \infty$, $0 < T \leq \infty$. Let $f(u)$ be continuously differentiable and let $f'(u)$ be bounded. Let u and w satisfy

$$\frac{\partial^2 w}{\partial x^2} + c\frac{\partial w}{\partial x} + f(w) - \frac{\partial w}{\partial t} \leq \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} + f(u) - \frac{\partial u}{\partial t}$$

in D, where c is a constant, and

$$w(0,x) \geq u(0,x) \qquad x \in (a,b)$$

if $a > -\infty$: $\quad w(t,a) \geq u(t,a) \qquad t \in [0,T)$

if $b < \infty$: $\quad w(t,b) \geq u(t,b) \qquad t \in [0,T)$

Then $w \geq u$ in D and if $w(0,x) > u(0,x)$ in an open sub-interval of (a,b), then $w > u$ in D.

And in exactly the same spririt we also have

Let $f: \mathbb{R} \to \mathbb{R}$ be C^1 with bounded derivative.

Theorem 6 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $T > 0$. Assume v and w are continuous functions on $[0,T] \times \bar{\Omega}$ with values in $\mathbb{R}$ that are ~~are~~ once differentiable wrt t and twice wrt x on $(0,T] \times \Omega$ and such that on $(0,T] \times \Omega$

$$\frac{\partial v}{\partial t} \leq \Delta v + f(v)$$

$$\frac{\partial w}{\partial t} \geq \Delta w + f(w)$$

$$v(t,x) \leq w(t,x) \quad \text{for } t=0, x \in \Omega \text{ and for } x \in \partial\Omega,\ 0 \leq t < T$$

Then

$$v(t,x) \leq w(t,x) \quad \text{for } (t,x) \in [0,T] \times \bar{\Omega}$$

Remark: f' needs only to be bounded on a subset of $\mathbb{R}$ that contains the range of both u and w

Applications of the Maximum- and Comparison Principle

Consider
$$\frac{\partial u}{\partial t} = \Delta u + f(u) \quad , \quad u\big|_{\partial\Omega} = 0$$
where $u(t,x) \in \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ (so we deal with a scalar equation). Let f be C^1. We associate with this equation a dynamical system on $C(\bar{\Omega})$, where as before $\bar{\Omega}$ denotes the closure of the bounded open subset Ω of $\mathbb{R}^n$.

Let ϕ_e be a steady state and let λ_d be the principal eigenvalue corresponding to the linearization at ϕ_e, with ψ_d the corresponding positive eigenfunction.

Theorem 7 $\lambda_d > 0 \Rightarrow \phi_e$ is unstable

Proof Since f' is continuous, there exists $\varepsilon > 0$ such that
$$|f'(\phi_e(x)+h) - f'(\phi_e(x))| < \tfrac{1}{2}\lambda_d \quad , \quad \forall x \in \bar{\Omega}$$
if $|h| < \varepsilon$. If ϕ_e is stable, we can associate to this ε a $\delta > 0$ such that for all $t \geq 0$
$$\sup_{x \in \bar{\Omega}} |u(t,x) - \phi_e(x)| < \varepsilon$$
provided
$$\sup_{x \in \bar{\Omega}} |u(0,x) - \phi_e(x)| < \delta$$
Let $u(0,x)$ satisfy this bound and, in addition, "lie above ϕ_e" by which we mean that
$$\phi_e(x) < u(0,x) < \phi_e(x) + \delta \quad , \quad \forall x \in \Omega$$

Theorem 6 implies that $u(t,x) \geq \phi_e(x)$ in $\bar{\Omega}$.

Now define

$$g(t) = \int_\Omega \big(u(t,x) - \phi_e(x)\big)\,\psi_d(x)\,dx$$

then $g(0) > 0$ and

$$g'(t) = \int_\Omega \frac{\partial u}{\partial t}\psi_d = \int_\Omega (\Delta u + f(u))\psi_d = \int_\Omega \big(\Delta(u-\phi_e) + f(u) - f(\phi_e)\big)\psi_d$$

$$= \int_\Omega (u-\phi_e)\Big[\Delta\psi_d + \frac{f(u)-f(\phi_e)}{u-\phi_e}\psi_d\Big] = \int_\Omega (u-\phi_e)\Big[\lambda_d - f'(\phi_e) + \frac{f(u)-f(\phi_e)}{u-\phi_e}\Big]\psi_d$$

The mean value theorem and our choice of ε together imply that

$$\left|\frac{f(u)-f(\phi_e)}{u-\phi_e} - f'(\phi_e)\right| < \tfrac{1}{2}\lambda_d$$

and since both $u - \phi_e \geq 0$ and $\psi_d \geq 0$ we find that

$$g'(t) \geq \tfrac{1}{2}\lambda_d\, g(t)$$

and hence $g(t) \geq g(0)\,e^{\frac{1}{2}\lambda_d t} \to \infty$ for $t \to \infty$ while in fact $g(t) < \varepsilon \int_\Omega \psi_d(x)\,dx$. So the assumption that ϕ_e is stable leads to a contradiction. $\square$

See the paper by Matano listed on page 9.18 for a far more wide ranging analysis of the "one-sided" instability of steady states of scalar equations, also in the case of no-flux boundary conditions.

9.28/ Next let's turn to the important topic of ~~sb~~ sub- and supersolutions: we call $\psi \in C^2(\Omega) \cap C(\bar{\Omega})$ a

supersolution if $\Delta\psi + f(\psi) \leq 0$, $\psi|_{\partial\Omega} \geq 0$

subsolution if $\Delta\psi + f(\psi) \geq 0$, $\psi|_{\partial\Omega} \leq 0$

but note that supersolutions are also called upper solutions and subsolutions are also ~~sto~~ called lower solutions.

Theorem 8 If ψ is a supersolution then the solution of

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \quad u|_{\partial\Omega} = 0$$

with initial condition $u(0,x) = \psi(x)$, $x \in \Omega$,

is a monotonically nonincreasing function of t. Likewise, if ψ is a subsolution, u is nondecreasing.

Proof Let ψ be a supersolution. By Theorem 6 we find that $u(t,x) \leq \psi(x)$, $x \in \bar{\Omega}$, $t \geq 0$. The function $\tilde{u}(t,x) = u(t+h,x)$, with $h > 0$, satisfies the equation with initial data $\tilde{u}(0,x) = u(h,x) \leq \psi(x)$, $x \in \bar{\Omega}$

so once again by Theorem 6 we obtain that

$$\tilde{u}(t,x) \leq u(t,x)$$

which means that u is nonincreasing in t. For ψ a subsolution the proof is, mutatis mutandis, the same $\square$

9.29/ The following result is intuitively obvious and yet the proof, which we omit, requires some care (one first proves that the limit is a weak solution and in a second step it is established that the weak solution is in fact a classical solution; see nonlineardiffusioncwisyllabus.pdf Theorem III.2.12, page 67).

<u>Lemma 9</u> If $u(t,x)$ is a uniformly bounded solution of

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \quad , \quad u|_{\partial\Omega} = 0$$

and

$$\lim_{t\to\infty} u(t,x) = \phi(x) \qquad \text{exists,}$$

then ϕ is a steady state, i.e., satisfies

$$\Delta\phi + f(\phi) = 0 \quad , \quad \phi|_{\partial\Omega} = 0$$

<u>Corollary 10</u> If $\underline{\Psi}$ is a subsolution, $\overline{\Psi}$ is a supersolution and these two are ordered, i.e.

$$\underline{\Psi}(x) \leq \overline{\Psi}(x) \quad , \quad x \in \overline{\Omega}$$

then there exists a steady state ϕ_e with

$$\underline{\Psi}(x) \leq \phi_e(x) \leq \overline{\Psi}(x) \quad , \quad x \in \overline{\Omega}$$

If there exists at most one steady state that is in between $\underline{\Psi}$ and $\overline{\Psi}$ in this sense, then any initial condition in between $\underline{\Psi}$ and $\overline{\Psi}$ is in the domain of attraction of ϕ_e (meaning that the solution of the initial value problem converges to ϕ_e for $t\to\infty$)

Exercise (inspired by Lecture Notes of C. Cosner)

Let $\Omega \subset \mathbb{R}^n$ be bounded and open.
Define $f(u) = u(1-u)$ and consider for $r>0$

$$\frac{\partial u}{\partial t} = \Delta u + r f(u) \qquad , x \in \Omega, t>0$$

$$u\big|_{\partial\Omega} = 0 \qquad , t>0$$

$$u(0,x) = u_0(x) \qquad , x \in \overline{\Omega}$$

Let λ_d be the principal eigenvalue of Δ with big monster boundary conditions on Ω, and let ψ_d be the corresponding eigenfunction. Note that $\lambda_d < 0$ and that $\psi_d(x) > 0$ for $x \in \Omega$.

i) Show that $0 \le u_0(x) \le 1, x \in \overline{\Omega} \Rightarrow 0 \le u(t,x) \le 1, x \in \overline{\Omega}$

ii) Consider a bounded and nonnegative initial condition u_0 and define $\bar{u}_0 = \sup_{x \in \Omega} u_0(x)$

and subsequently define $\bar{u}(t)$ as the solution of the ODE $\frac{d\bar{u}}{dt} = f(\bar{u})$ with initial condition $\bar{u}(0) = \bar{u}_0$.

Use Theorem 6 to prove that $u(t,x) \le \bar{u}(t)$, $\forall x \in \overline{\Omega}$

iii) We restrict our attention to nonnegative initial conditions and to nonnegative steady states. We call the steady state that is identically zero trivial. Show that for $r < -\lambda_d$ no nontrivial steady state can exist.

Hint on next page.

9.31 Hint: Assuming a nontrivial nonnegative steady state ϕ_e exists, multiply the equation for ϕ_e by Ψ_d and integrate by parts

iv) Show that the trivial steady state is globally asymptotically stable if $r < -\lambda_d$, in the sense that for every bounded nonnegative initial condition u_0 the solution converges to zero for $t \to \infty$.

v) Next consider the case $r > -\lambda_d$. Show that a nontrivial steady state exists.
Hint: show that $\varepsilon \Psi_d$ is for small $\varepsilon > 0$ a subsolution

vi) Show that $0 \leq \phi_e^{(1)} \leq \phi_e^{(2)}$ with $\phi_e^{(1)}$ and $\phi_e^{(2)}$ steady states, $\phi_e^{(1)} \neq \phi_e^{(2)}$, is impossible
Hint: remember the hint for iii) and do something similar

vii) Show that for $r > -\lambda_d$ there exists a unique nontrivial equilibrium ϕ_e which is globally asymptotically stable in the sense that for every nontrivial bounded nonnegative initial condition u_0 the solution converges to ϕ_e for $t \to \infty$. You are allowed to <u>assume</u> that every such solution is for $t > 0$ bounded below by $\varepsilon \Psi_d$ with ε sufficiently small.

9.32/ For more information you may consider:

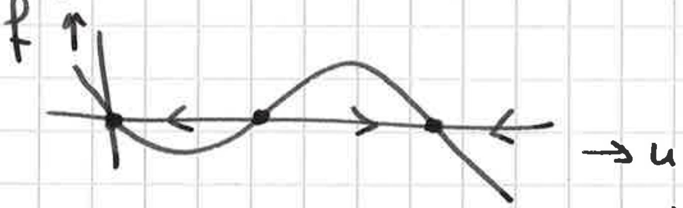
P. Polacik, Parabolic equations: asymptotic behavior and dynamics on invariant manifolds
pp. 835–883 in Handbook of Dynamical Systems, Vol. 2, B. Fiedler (ed.), Elsevier 2002

H. L. Smith, Monotone Dynamical Systems: an introduction to the theory of competitive and cooperative dynamical systems, AMS, 1995

We now ~~swithc~~ switch to the final topic concerning (in)stability of steady states:

Gradient Systems

For scalar ODE we can conclude that all bounded orbits converge to a steady state from pictures like



$\dot{u} = f(u)$

but alternatively we can introduce (the) a primitive F of f, i.e.,

$$F(u) = \int_0^u f(\sigma)\,d\sigma$$

and compute that the time derivative of $-F(u(t))$ along an orbit is

$$\frac{d}{dt}\left(-F(u(t))\right) = -f(u(t))\,\dot{u}(t) = -\left(f(u(t))\right)^2 \leq 0$$

with equality iff $u(t) = u_e$ with $f(u_e) = 0$

(the $-$ is a matter of convention; we call $-F$ a Lyapunov function and the convention is that Lyapunov functions decrease along orbits). So if $-F$ is bounded from below, the orbit must converge to a steady state.

This approach generalizes to a rather limited class of ODE systems of higher dimension, viz., those for which $f: \mathbb{R}^k \to \mathbb{R}^k$ is the <u>gradient</u> of a function $F: \mathbb{R}^k \to \mathbb{R}$.

Indeed, if
$$f_i(u) = \frac{\partial F}{\partial u_i}(u)$$
then
$$\frac{d}{dt}\left(-F(u(t))\right) = -\nabla F(u(t)) \cdot \dot{u}(t) = -\left|f(u(t))\right|^2 \leq 0$$
with equality iff $u(t) = u_e$ with $f(u_e) = 0$.

So steady states correspond to critical points of F. The stability is determined by the eigenvalues of the Jacobi matrix $\dfrac{\partial f_i}{\partial u_j} = \dfrac{\partial^2 F}{\partial u_i \partial u_j}$ so by the Hessian matrix of F. Since the Hessian matrix is symmetric, all eigenvalues are real. If all eigenvalues are negative, the matrix is negative definite and F has a maximum and, equivalently, $-F$ has a minimum. So minima of $-F$ correspond to stable steady states and if a critical point of $-F$ is a saddle point or even a maximum then the corresponding steady state is unstable.

Our next example is infinite dimensional. Consider
$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in } \Omega$$
$$u\big|_{\partial\Omega} = 0$$

9.34/ and the (quadratic) functional $V: H_0^1(\Omega) \to \mathbb{R}$ defined by

$$V(\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2$$

We (formally) compute the time derivative of $V(u(t))$ where $u(t)(x) = u(t,x)$ with u the solution of the linear parabolic PDE:

$$\frac{d}{dt} V(u(t)) = \frac{1}{2}\sum_i \frac{d}{dt}\int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^2 = \sum_i \int_\Omega \frac{\partial^2 u}{\partial x_i \partial t}\frac{\partial u}{\partial x_i}$$

$$= -\int_\Omega \frac{\partial u}{\partial t}\sum_i \frac{\partial^2 u}{\partial x_i^2} = -\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 \leq 0$$

with equality iff $\frac{\partial u}{\partial t}$ is identically zero. Here we have taken for granted that the mixed derivative $\frac{\partial^2 u}{\partial x_i \partial t}$ exists and that, when we integrate by parts, the boundary term involving $\frac{\partial u}{\partial t}\frac{\partial u}{\partial n}$ vanishes since $u|_{\partial\Omega}$ is not depending on time.

Side remark

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(V(\phi+\varepsilon\psi) - V(\phi)\right) = \int_\Omega \nabla\phi\cdot\nabla\psi$$

$$= -\int_\Omega \psi\,\Delta\phi$$

so we can interpret $\Delta u = -DV(u)$, so as a gradient. (But a complicated aspect of a further precise elaboration is that, in order to keep viewing H_0^1 as a subspace of L_2, we need to represent its dual space by H^{-1} (distributional derivatives of L_2-functions) and not exploit that H_0^1 is a Hilbert space, so "is" its own dual space.)

9.35) In the rest of this "chapter" we consider

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) \quad , 0<x<1 , t>0$$

$$u(t,0) = 0 = u(t,1) \quad \text{or} \quad \frac{\partial u}{\partial x}(t,0) = 0 = \frac{\partial u}{\partial x}(t,1)$$

$$u(0,x) = \phi(x)$$

with $\phi \in C[0,1]$ and, in case of the big monster boundary conditions, $\phi(0)=0=\phi(1)$. We refer to nonlineardiffusioncwi syllabus.pdf Chapter V for precise formulations and here only sketch the main ideas.

The fact that the equation "smoothes" the dependence on the variable x is important and therefore we also introduce the space $C^1[0,1]$ with, in the case of no-flux boundary conditions, $\phi'(0)=0=\phi'(1)$. So in the following we use the somewhat debatable notation:

	big monster	no-flux
C	$\phi(0)=0=\phi(1)$	
C^1		$\phi'(0)=0=\phi'(1)$

where all functions are defined on $[0,1]$, C^1 is always a subspace of C, on C we have the supremum norm and on C^1 we take as the norm the sum of the supremum norm of the function itself and the supremum norm of the derivative. By the Arzela-Ascoli Theorem, C^1 is compactly embedded in C.

9.36/ A key point is that if the initial function ϕ belongs to C, for every $t>0$ the function $u(t,\cdot)$ belongs to C^1.

Define $V: C^1 \to \mathbb{R}$ by

$$V(\phi) = \int_0^1 \left(\frac{1}{2}(\phi'(x))^2 - F(\phi(x)) \right) dx$$

where, as before, $F(u) = \int_0^u f(\sigma)\, d\sigma$.

<u>Theorem 11</u>
$$\frac{d}{dt} V(u(t,\cdot)) = -\int_0^1 \left(\frac{\partial u}{\partial t}\right)^2$$

<u>Theorem 12</u> If V attains a strict local minimum in an equilibrium point ϕ_e, then ϕ_e is stable and if ϕ_e is isolated it is asymptotically stable.

<u>Theorem 13</u> Let ϕ_e be a steady state and let λ_d be the principal eigenvalue corresponding to the linearization in ϕ_e. Then

$$\lambda_d < 0 \Rightarrow \phi_e \text{ is asymptotically stable}$$