Lecture 5 : finding steady states of scalar reaction-diffusion equations on an interval by way of phase plane analysis of a Hamiltonian system

<u>Starting point</u> $\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + r f(u) \qquad 0 < x < L$

either $u(t,0) = 0 = u(t,L)$ or $\frac{\partial u}{\partial x}(t,0) = 0 = \frac{\partial u}{\partial x}(t,L)$

and $f(u) = u g(u)$ with $g(0) = 1$ (so r corresponds to the net per capita growth rate in the low density limit and g incorporates the density dependence).

By scaling of x and t we can, as before, achieve that $L = 1$ and $d = 1$. Then <u>steady states</u> are solutions of the <u>nonlinear</u> BVP (Boundary Value Problem)

$$(BVP) \begin{cases} \phi'' + r f(\phi) = 0 \qquad 0 < x < 1 \\ \text{either } \phi(0) = 0 = \phi(1) \; ② \; \text{ or } \phi'(0) = 0 = \phi'(1) \; ① \end{cases}$$

If we ignore for the time being the boundary conditions, we can write the second order equation as a system of two first order equations

$$(ODE) \begin{cases} u' = v \\ v' = -r f(u) \end{cases}$$

In this context, we usually call the independent variable "time" (even though it is in fact the spatial position) and investigate the phase portrait that depicts the orbit structure of the dynamical system generated by these two ODE

5.2/ (even though this has nothing to do with the dynamical system on a space of functions of x generated by the reaction-diffusion equation).

We can rewrite (ODE) in the form

$$(\text{ODE}) \begin{cases} u' = \dfrac{\partial H}{\partial v} \\ v' = -\dfrac{\partial H}{\partial u} \end{cases}$$

where the Hamiltonian H is defined by

$$H(u,v) = \tfrac{1}{2} v^2 + r F(u)$$

with F a primitive of f or, more precisely $F(u) = \int_0^u f(\sigma) d\sigma$

The point is that along orbits

$$\frac{d}{dt} H(u(t), v(t)) = \frac{\partial H}{\partial u} u' + \frac{\partial H}{\partial v} v' = 0$$

so orbits are restricted to level sets of H.

Clearly $H(u,v) = H(u,-v)$, so reflection in the u-axis maps level sets to level sets. And, therefore, orbits to orbits. But note that the direction in which the level set is traversed changes: u increases if $v>0$ and decreases if $v<0$.

Equilibria of (ODE) lie on the u-axis and are determined by $f(u) = 0$, so correspond to the zero's of f. (In the case of no-flux boundary conditions, these equilibria of (ODE) yield constant (as a function of x) steady states of the reaction-diffusion equation since they are indeed constant solutions of (BVP); in the case of the big monster, this is only true for the equilibrium $(0,0)$ of (ODE))

5.3 Let $\bar{u}$ be such that $f(\bar{u})=0$. If we linearize (ODE) at the equilibrium $(\bar{u},0)$ we obtain

$$(\text{LODE}) \quad \begin{cases} w' = v \\ v' = -r f'(\bar{u}) w \end{cases}$$

where $w = u - \bar{u}$ and higher order terms have been deleted.

To this linear system corresponds the quadratic Hamiltonian

$$\tfrac{1}{2} v^2 + \tfrac{1}{2} r f'(\bar{u}) w^2$$

The matrix $\begin{pmatrix} 0 & 1 \\ -rf'(\bar{u}) & 0 \end{pmatrix}$ corresponding to (LODE)

has eigenvalues that are found by solving the characteristic equation $\lambda^2 + r f'(\bar{u}) = 0$

Excluding the degenerate case $f'(\bar{u}) = 0$ (for which one has to investigate the higher order terms in order to determine the local phase portrait), we distinguish two cases:

Case 1 $f'(\bar{u}) > 0 \Rightarrow \lambda = \pm i \sqrt{r f'(\bar{u})}$ and H has a minimum in $(\bar{u},0)$. So $(\bar{u},0)$ is a center surrounded by a family of closed orbits (corresponding to periodic solutions of (ODE))

Case 2 $f'(\bar{u}) < 0 \Rightarrow \lambda = \pm \sqrt{-r f'(u)}$ and H has a saddle point in $(\bar{u},0)$. The equilibrium of (ODE) in $(\bar{u},0)$ is also a saddle point in the dynamical systems sense.

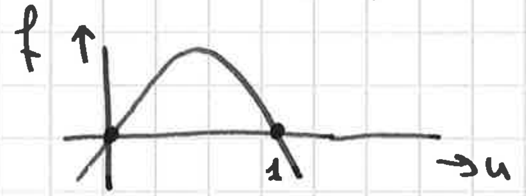
5.4 Correspondence ~~Connections~~ between pieces of orbits of (ODE) and solutions of (BVP):

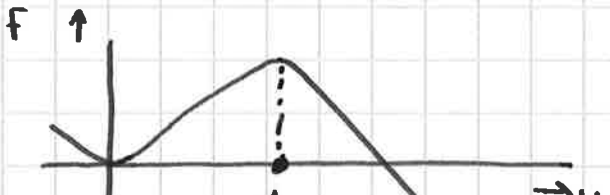
① A solution of $(BVP)_{\text{no-flux}}$ corresponds to a piece of an orbit of (ODE) that connects a point on the u-axis to another, or the same, point on the u-axis and that takes "time" (i.e., an interval of values of the independent variable of length) 1 to ~~transverse~~ traverse

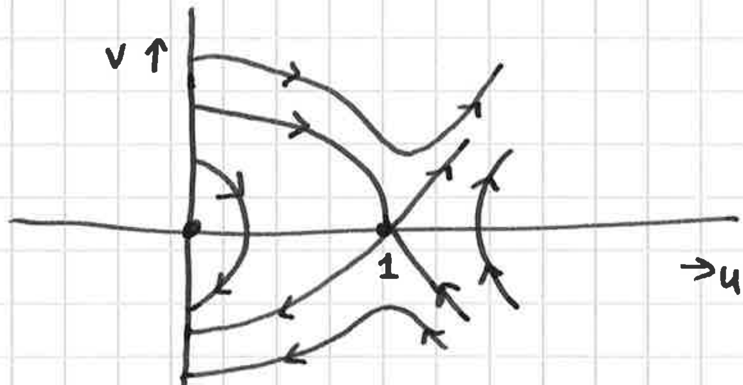
② A solution of $(BVP)_{\text{big monster}}$ corresponds to a piece of an orbit of (ODE) that connects a point on the v-axis to another, or the same, point on the v-axis and that takes "time" 1 to ~~transverse~~ traverse

<u>First Example</u> Logistic growth $f(u) = u\left(1 - \frac{u}{K}\right)$

By scaling of u we achieve that $K = 1$.





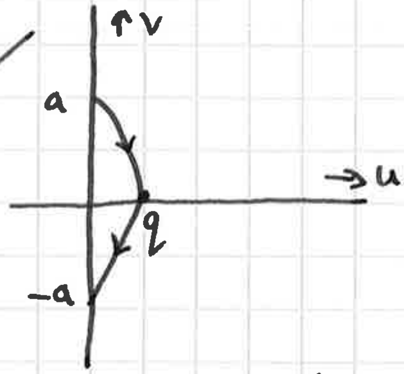


We restrict to $u \geq 0$ because u represents a population density

① only $(1,0)$ qualifies and it corresponds to

$\phi(x) = 1 \quad 0 \leq x \leq 1$

5.5



② The depicted orbit piece is a candidate, but if it takes "time" T_q to travel from a to $-a$, we need to have that $T_q = 1$.

How to compute T_q ?

Since the Hamiltonian H is constant along the orbit, we know that

$$H(u,v) = H(q,0) \quad \text{or} \quad \tfrac{1}{2} v^2 + r F(u) = r F(q)$$

Let $(u(x), v(x))$ denote the solution of (ODE) with initial condition $(u(0), v(0)) = (0, a)$

The reflection symmetry $H(u,v) = H(u,-v)$ tells us that $\left(u(T_q - x), v(T_q - x)\right) = \left(u(x), -v(x)\right)$, $0 \leq x \leq T_q$

In particular, it takes as long to travel from a to q as it takes to travel from q to a.

(ODE) tells us that $v = \frac{du}{dx}$ and if we insert this into $\frac{1}{2} v^2 + r F(u) = r F(q)$ we find that between a and q, where $\frac{du}{dx} \geq 0$, we have

$$\frac{du}{dx} = \sqrt{2r\left(F(q) - F(u)\right)}$$

Hence

$$T_q = 2 \int_0^q \frac{dx}{du}\, du = \sqrt{\frac{2}{r}} \int_0^q \frac{du}{\sqrt{F(q) - F(u)}}$$

$$= \sqrt{\frac{2}{r}} \int_0^1 \frac{q\, d\eta}{\sqrt{F(q) - F(q\eta)}} =: \sqrt{\frac{2}{r}}\, G(q)$$

5.6/ With $f(u) = ug(u)$ we have that

$$F(q) - F(q\eta) = \int_{q\eta}^{q} \sigma g(\sigma)\, d\sigma = q^2 \int_{\eta}^{1} \tau g(q\tau)\, d\tau$$ and hence

$$T_q = \sqrt{\frac{2}{r}} \int_0^1 \frac{d\eta}{\left(\int_\eta^1 \tau g(q\tau)\, d\tau\right)^{1/2}}$$

<u>Lemma</u> If g is decreasing then $q \mapsto T_q$ is increasing

<u>Lemma</u> $T_q \to \infty$ for $q \uparrow 1$

(the amount of time that the orbit spends in the neighbourhood of the saddle point $(1,0)$ increases without upperbound when we consider orbits that come closer and closer to the saddle point; analytically we have that $\int_{1-\varepsilon}^{1} \tau g(\tau)\, d\tau = -\frac{1}{2} g'(1)\varepsilon^2 + o(\varepsilon^2)$ for $\varepsilon \downarrow 0$
(recall that $g(1) = 0$)

So for $q = 1$ the denominator of the integrand behaves like ε when $\eta = 1 - \varepsilon$ and $\varepsilon \downarrow 0$. So we have, in the limit $q \uparrow 1$, a non-integrable singularity. Hence $T_q \to \infty$)

<u>Lemma</u> $T_0 = \frac{\pi}{\sqrt{r}} \leftarrow \frac{\pi}{\sqrt{r}}$ $\quad \left(\mathcal{G}(0) = \frac{\pi}{\sqrt{2}} \right)$

<u>Proof</u> Since $g(0) = 1$ we have $T_0 = \sqrt{\frac{2}{r}} \int_0^1 \frac{d\eta}{\left(\int_\eta^1 \tau\, d\tau\right)^{1/2}}$

$$= \frac{2}{\sqrt{r}} \int_0^1 \frac{d\eta}{\sqrt{1-\eta^2}} = \frac{2}{\sqrt{r}} \int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin^2 x}}\, dx = \frac{\pi}{\sqrt{r}} \qquad \square$$

5.7

Theorem Let q be decreasing. Then T_q increases from $\frac{\pi}{\sqrt{r}}$ for $q=0$ to infinity for $q \uparrow 1$

Corollary $(BVP)_{\text{big monster}}$ has no positive solution for $r \leq \pi^2$. For any $r > \pi^2$ it has a unique positive solution $\phi_r = \phi_r(x)$. This solution is an increasing function of x for $0 \leq x < \frac{1}{2}$ and satisfies

$$\phi_r(1-x) = \phi_r(x)$$

Proof Solutions of $(BVP)_{\text{big monster}}$ that are positive are in one-to-one correspondence with solutions of the equation $T_q = 1$, with q corresponding to $\phi(\frac{1}{2})$ □

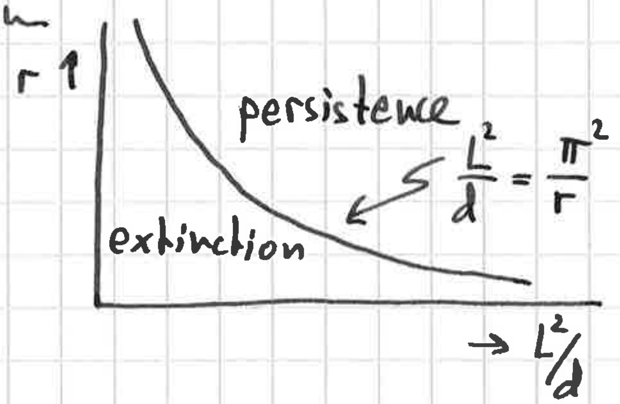
Exercise i) Prove that $\phi_r(x)$ is an increasing function of r.

ii) Prove that for $x \in (0,1)$ we have $\phi_r(x) \uparrow 1$ for $r \to \infty$.

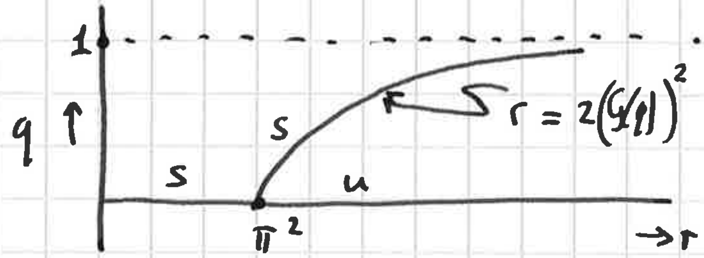
iii) Recalling the scaling of page 5.1, interpret the condition $r > \pi^2$ in terms of minimal patch size

iv) In terms of the original (i.e., not yet scaled) parameters we have the diagram

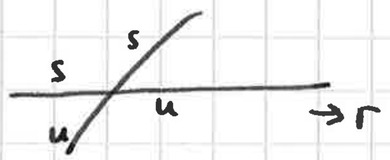
What is the dimension of the variables along the axes?



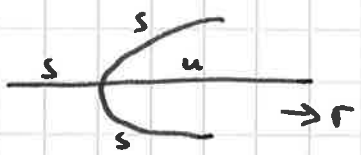
5.8/ We can also summarize the results in a so-called <u>bifurcation diagram</u>:



Here the labels s and u refer to, respectively, stable and unstable. (But we haven't yet explained how to obtain this information about the stability character of the various steady states of the parabolic problem that was our starting point on pages 5.1, so at this point these labels only serve to make you curious about the methods to justify them.)
Because we restricted our attention to positive solutions, we cannot yet decide whether the local picture near $r = \pi^2$ looks like



or like



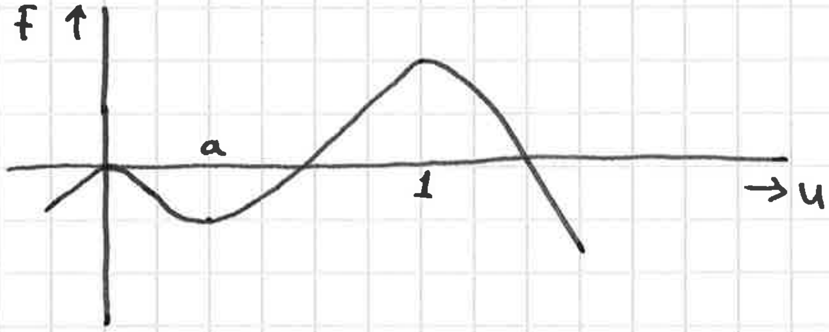
In the first case we speak about a <u>transcritical bifurcation</u> and in the second case about a <u>pitchfork bifurcation</u>. In a later chapter we shall explain how one can, for much more general problems, compute the local features of the bifurcation diagram near an eigenvalue zero of the linearized problem (like here for $r = \pi^2$)

5.9/ **Second Example** Bistability, e.g., $f(u) = u(u-a)(1-u)$

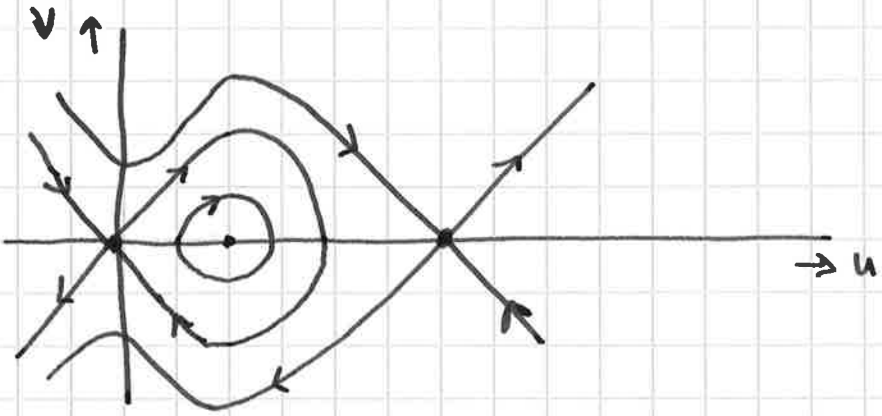
NB symbol a here has no relationship to a in picture on p. 5.5

Pictures are for the case

$$F(1) = \int_0^1 f(\sigma)\,d\sigma > 0$$



If $F(1) < 0$ there is a homoclinic loop from the saddle point $(1,0)$



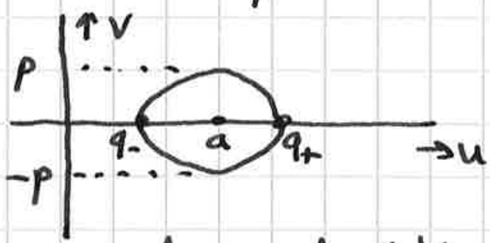
Note that stable- and unstable manifolds of saddle points form an obstruction for orbits that connect two points on the u-axis or two points on the v-axis.

So candidates for ~~ste~~ solutions of (BVP) ~~are~~ with zero flux boundary conditions are obtained by making (k or) $k+\frac{1}{2}$ turns along a closed orbit surrounding the equilibrium $(a,0)$ of the Hamiltonian system.

Exercise Describe the orbits that yield candidates for solutions of $(BVP)_{\text{big monster}}$

Exercise Check that one can eliminate the parameter r from (ODE) by scaling v with a factor $\sqrt{r}$ and time by a factor $1/\sqrt{r}$

Motivated by the last exercise, we now first focus on the special case $r=1$ and then bring the parameter r back in when we attempt to satisfy the boundary conditions of (BVP).



A closed orbit around $(a,0)$ can be characterized in various ways. We denote the maximum and the minimum of v by, respectively, p and $-p$. And the maximum and the minimum of u by, repectively, q_+ and q_-. Let H denote the value of the Hamiltonian $H(u,v) = \frac{1}{2}v^2 + F(u)$ (recall that $r=1$ for the time being) on the orbit. Then

(i) $F(q_+) = F(q_-)$ (recall $F(u) = \int_0^u f(\sigma)d\sigma$)

(ii) $H = \frac{1}{2}p^2 + F(a) = F(q_\pm)$

We choose q_-, with $0 < q_- < a$, as the parameter to describe the one-parameter family of closed orbits.
We denote the period by T_{q_-}. Then

$$T_{q_-} = \sqrt{2}\int_{q_-}^{q_+} \frac{d\sigma}{\sqrt{F(q_\pm) - F(\sigma)}}$$

Exercise Use arguments like those described on page 5.6 to deduce that

$$\lim_{q_-\downarrow 0} T_{q_-} = \infty$$

$$\lim_{q_-\uparrow a} T_{q_-} = \frac{2\pi}{\sqrt{f'(a)}}$$

So the range of the continuous function $q_- \mapsto T_{q_-}$ is unbounded and every number that exceeds $2\pi/\sqrt{f'(a)}$ is included. It is very hard to determine for a given function f whether or not $q_- \mapsto T_{q_-}$ is monotone decreasing, see C. Chicone, The monotonicity of the period function for planar Hamiltonian vector fields, J. Diff. Equ. (1987) 69: 310–321 and the references given there.

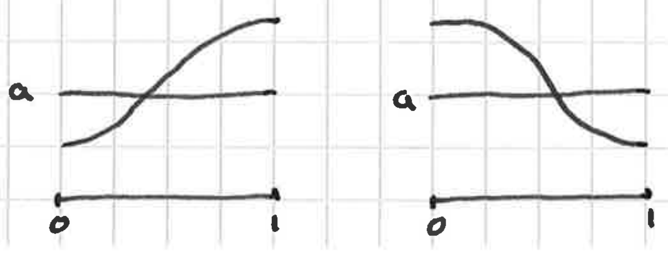
Now let's re-introduce the parameter r. The effect is that the v-axis needs stretching, that p should be replaced by $\sqrt{r}\,p$ and H by rH, but concerning the u-axis nothing changes and in particular we still parameterize the family of closed orbits by q_- with $0<q_-<a$. The effect is also that T_{q_-} should be replaced by $T_{q_-}/\sqrt{r}$. In particular we have: both the "half-orbit" connecting $(q_-,0)$ to $(q_+,0)$ and the "half-orbit" connecting $(q_+,0)$ to $(q_-,0)$ yield a solution of $(BVP)_{no-flux}$ with parameter r iff

$$\frac{1}{2}\, T_{q_-} \frac{1}{\sqrt{r}} = 1$$

Since we do not know whether or not the map $q_- \mapsto T_{q_-}$ is monotone, we cannot easily solve this scalar equation for q_- as a function of r. But it is straightforward to solve for r as a function of q_-:

$$r = \frac{1}{4}\left(T_{q_-}\right)^2$$

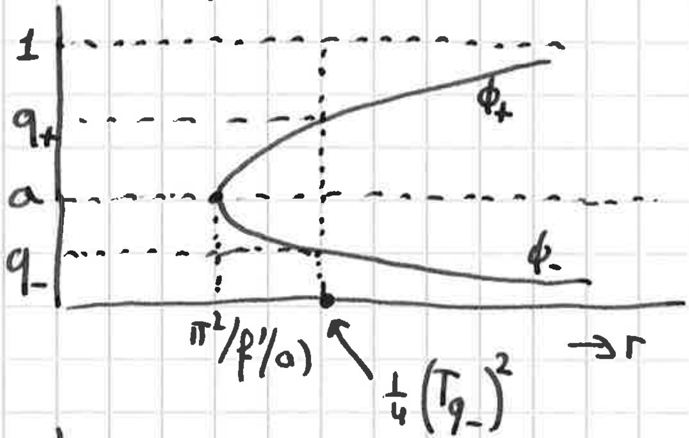
<u>Exercise</u> Check that the graphs of the two solutions of $(BVP)_{no-flux}$ look like



Let's call the one with an increasing graph ϕ_+ and the one with a decreasing graph ϕ_-

5.12 Check that $\phi_+(1-x) = \phi_-(x)$. Also check that the orbit with parameters q_- and r has period 2 and that ϕ_+ is obtained from ϕ_- by a shift over half the period and that, likewise, ϕ_- is obtained from ϕ_+ by a shift over half the period.

If we depict the relationship $\frac{1}{2} T_{q_-} \frac{1}{\sqrt{r}} = 1$ graphically, the two solutions are "on top of eachother". If, instead of q_-, we depict the value of ϕ in the right end point (so q_+ for ϕ_+ and q_- for ϕ_-) we get a more informative graph. If $q_- \mapsto T_{q_-}$ is monotone this graph looks like



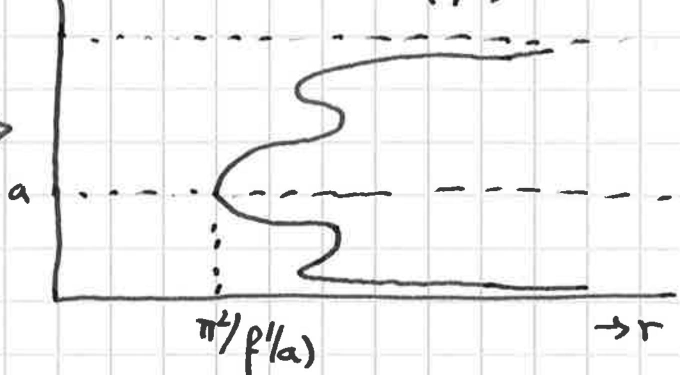
where q_+ and q_- are related to each other by

$$F(q_+) = F(q_-)$$

(note that for $q_- \uparrow a$ we have

$$q_+ - a = a - q_- + o(a - q_-)\ )$$

An example of a (q_-, r) relationship with non-monotone $q_- \mapsto T_{q_-}$



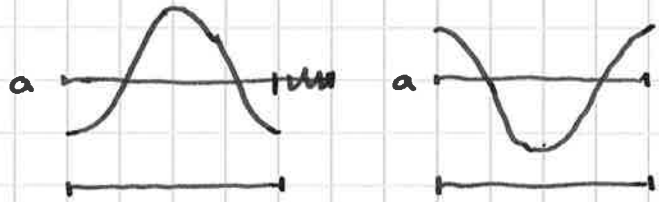
The next thing to notice is that we can, in an almost literal sense, recycle solutions by making k half turns, $k \geq 2$, rather than just one. So we require

$$\frac{k}{2} T_{q_-} \frac{1}{\sqrt{r}} = 1$$

leading to

$$r = \frac{k^2}{4} \left(T_{q_-}\right)^2$$

and to pairs of solutions that assume the value a exactly k-times



ϕ_{2+} ϕ_{2-}

Exercise Check that both ϕ_{2k+} and ϕ_{2k-} are mid-point-reflection symmetric and that one is obtained from the other by a shift over half the period, so over $1/2k$.
How are $\phi_{(2k+1)+}$ and $\phi_{(2k+1)-}$ related to each other?

Even though it doesn't make much biological sense, we can of course mathematically consider the boundary value problem

$$\phi'' + r f(\phi) = 0$$
$$\phi(0) = a = \phi(1)$$

In terms of the figure on page 5.10 we are then, first of all, interested in the "half-orbit" connecting (a,p) to $(a,-p)$ and the "half-orbit" connecting $(a,-p)$ to (a,p). In general the time T_{down} involved in the first is not equal to the time involved in T_{up}. The two equations

$$\frac{1}{\sqrt{r}} T_{down} = 1 \quad \text{and} \quad \frac{1}{\sqrt{r}} T_{up} = 1$$

have to be considered separately and, even though $T_{down} + T_{up} = T_{q_-}$, the value of q_- corresponding to the solution of one of these equations is not the value of q_- corresponding to the solution of the other of the two. Both solutions are symmetric with respect to mid-point reflection.

If we look for a solution corresponding to a full turn along the closed orbit, we have to satisfy

$$\frac{1}{\sqrt{r}} T_{q_-} = 1$$

and once we satisfy this equation we have two solutions, one corresponding to a full turn starting and ending in (a, p) and one to a full turn starting and ending in $(a, -p)$. Now one is obtained from the other by reflection in the mid-point. Both assume the value a exactly once in $(0,1)$, but not necessarily in $x = \frac{1}{2}$. One is also obtained from the other by a shift over the distance to the point in the interior where the value a is assumed(*). Both are also shifted versions of a solution of $(BVP)_{no-flux}$

The descriptions above of solutions corresponding to k half-turns with, respectively, $k=1$ and $k=2$ ~~des~~ apply to general k with, respectively, k odd and k even.

<u>Exercise</u> Consider T_{down} and T_{up} as functions of q_-. Show that

$$\lim_{q_- \uparrow a} T_{down} = \lim_{q_- \uparrow a} T_{up} = \frac{\pi}{\sqrt{f'(a)}}$$

Conclude that the solution $\phi \equiv a$ undergoes a bifurcation for ~~\$~~ $r = \frac{\pi^2}{f'(a)}$. Do you agree that we can construct the bifurcation diagram by plotting

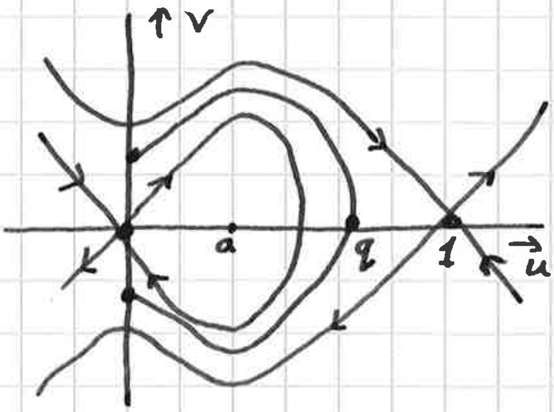
$r = \left(T_{down}(q_+)\right)^2$, $a < q_+ < q_+^{max}$ and $r = \left(T_{up}(q_-)\right)^2$, $0 < q_- < a$

while relating q_+ to q_- via $F(q_+) = F(q_-)$?

(*) This statement is not entirely correct, since one has to adjust the value of r.

5.15 Now let's look at (BVP)$_{\text{big monster}}$

The answer to the exercise at the bottom of page 5.9 is, with reference to this figure:



the orbits parameterized by q with $q_+^{max} < q < 1$

Let T_q denote the time it takes to arrive at the negative v-axis while starting at the positive v-axis and passing through $(q,0)$. Then, as on page 5.5, we have

$$T_q = \sqrt{\tfrac{2}{r}}\; G(q)$$

with

$$G(q) = \int_0^q \frac{du}{\sqrt{F(q)-F(u)}} = \int_0^1 \frac{q\,d\eta}{\sqrt{F(q)-F(q\eta)}}$$

By the arguments described on page 5.6 we deduce that

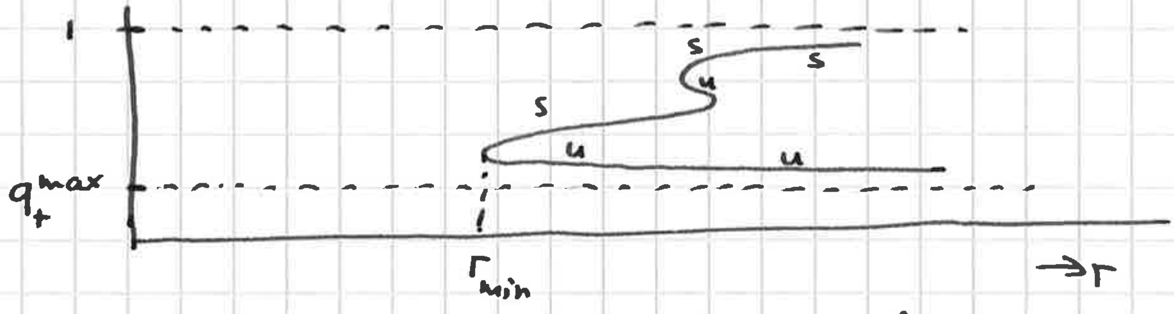
$$\lim_{q\downarrow q_{max}} G(q) = \infty = \lim_{q\uparrow 1} G(q)$$

The equation $T_q = 1$ amounts to $\quad r = 2\left(G(q)\right)^2$

Let G_{min} denote the minimum of $\{G(q) : q_+^{max} < q < 1\}$.

For every $r > r_{min} = 2\left(G_{min}\right)^2$ there are at least two values of q such that $T_q = 1$ and if $q \mapsto G(q)$ has no local minima other than the global minimum there are <u>exactly</u> two solutions for $r > r_{min}$ (in the book

5.16) J. Smoller, Shock waves and reaction-diffusion equations, Springer, 1983, you can find the proof that for $f(u) = u(u-a)(1-u)$ the function G has indeed precisely one minimum). In general, the solution set looks like



where once again the labels are yet to be justified. The figure depicts three so-called <u>saddle-node bifurcations</u> where the number of solutions changes by two when the parameter crosses the bifurcation value.

<u>Conclusion</u> By relating solutions of (BVP) to pieces of orbits (of (ODE)) that connect the v-axis to the v-axis (in case of the big monster) or the u-axis to the u-axis (in case of no-flux boundary conditions), we can obtain quite detailed global (in the parameter r) information about the set of steady state solutions of $\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + r f(u)$, $0 < x < L$, provided with either zero Dirichlet (= big monster) or zero Neumann (= no flux) boundary conditions.