3.1 Linear diffusion equation: find explicit expressions for solutions by superposition of elementary building blocks (recall Fundamental Solution, p. 2.4, property v: $u(t,x) = \int_{\mathbb{R}^n} \Phi(t,x-y)\, u_0(y)\, dy$)

Today: find building blocks by separation of variables

Example that you know very well: linear ODE

$$\frac{du}{dt} = Mu \;, \quad u \in \mathbb{R}^n, \quad M \in \mathbb{R}^{n \times n}$$

Ansatz $u(t) = a(t)\,\phi$ with $a(t) \in \mathbb{R}$ and $\phi \in \mathbb{R}^n$

$a(t)$: time dependence; ϕ: dependence on index $i = 1, \dots, n$

$$a'(t)\,\phi = a(t) M \phi \xRightarrow{\text{if } a(t) \neq 0} \frac{a'(t)}{a(t)}\,\phi = M\phi$$

rhs is independent of t, so the quotient $\frac{a'(t)}{a(t)}$ must be a constant, say λ

$$\Rightarrow a(t) = e^{\lambda t} \;\&\; M\phi = \lambda\phi$$

where the last equation determines the possible λ's

Assume: eigenvectors of M form a basis for $\mathbb{R}^n$ (note: this is true if M is symmetric)

Then we can solve $\dot{u} = Mu$, $u(0) = u_0$ by finding $c_1, \dots, c_n$ such that $u_0 = \sum_{i=1}^n c_i \phi_i$ and next putting $u(t) = \sum_{i=1}^n c_i e^{\lambda_i t} \phi_i$

so by expressing the general solution as a linear combination of the building blocks $e^{\lambda_i t}\phi_i$

3.2/ For large t, the building block(s) for which the real part of λ is maximal matters most. In particular, we find exponential decay if $\mathrm{Re}\,\lambda_i < 0$ for $i = 1, \dots, n$ and exponential growth if for some index j we have $\mathrm{Re}\,\lambda_j > 0$.

For symmetric matrices all eigenvalues are real, but in general they can be complex (so we need to complexify, i.e., extend $M: \mathbb{R}^n \to \mathbb{R}^n$ to $M: \mathbb{C}^n \to \mathbb{C}^n$ via $M(x+iy) = Mx + iMy$, $x, y \in \mathbb{R}^n$). If M has real entries, complex eigenvalues occur in complex conjugate pairs. The linear combination $c e^{\lambda t} \phi + \bar{c} e^{\bar{\lambda} t} \bar{\phi}$, $c \in \mathbb{C}$, is then a building block for real solutions.

Now consider

$$(5.2.1) \qquad \frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + ru$$

$$(5.2.9) \qquad u(t,0) = 0 = u(t,L)$$

$$(5.2.6) \qquad u(0,x) = u_0(x)$$

involving three parameters, the diffusion coefficient d, the per capita net growth rate r and the length L of the interval to which individuals are restricted. We claim that without loss of generality we can take $d = 1$ and $L = 1$. To substantiate this claim, we perform <u>scaling</u>

$$\xi = \frac{x}{L} \qquad t = \frac{L^2}{d} \tau \qquad r_{new} = \frac{r_{old} L_{old}^2}{d_{old}}$$

<u>Exercise</u> Elaborate the details

After the scaling we recycle the symbols and use again x and t to denote the scaled variables.

3.3 So we consider $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + ru$

$u(t,0) = 0 = u(t,1)$

and make the

Ansatz $u(t,x) = a(t)\,\phi(x)$

$$\Rightarrow \frac{a'(t)}{a(t)}\phi(x) = \cancel{a(t)}\phi''(x) + r\cancel{a(t)}\phi(x)$$

λ

$\Rightarrow a(t) = e^{\lambda t}$ and

(λ,ϕ) should be a solution of the <u>eigenvalue problem</u>

$$\begin{cases} \phi'' + r\phi = \lambda\phi \\ \phi(0) = 0 = \phi(1) \end{cases}$$

Note that if ϕ is a solution, so is $\tilde{\phi}$ defined by $\tilde{\phi}(x) = \phi(1-x)$ (in more fancy jargon: the problem is equivariant with respect to <u>reflection</u> in the midpoint)

Also note the a priori bound $\lambda < r$ [Indeed, if we multiply the equation for ϕ by ϕ and integrate over $[0,1]$ we obtain, by partial integration, the identity

$$-\int |\phi'|^2 + r\int |\phi|^2 = \lambda \int |\phi|^2$$

which shows that necessarily $\lambda < r$]

So the building blocks for ϕ itself are $e^{\pm i\sqrt{r-\lambda}\,x}$

(recall the theory of second order linear differential equations with constant coefficients)

3.4 In order to satisfy the boundary condition $\phi(0)=0$ we focus on the linear combination $\sin(\sqrt{r-\lambda}\, x)$

The boundary condition $\phi(1)=0$ now acts as a condition on λ, viz., $\sin(\sqrt{r-\lambda})=0$

and hence $\sqrt{r-\lambda} = k\pi, \ k=1,2,\dots$

or
$$\lambda = \lambda_k = r-(k\pi)^2 \qquad k=1,2,\dots$$

and
$$\phi_k(x) = \sqrt{2}\, \sin k\pi x$$

where the factor $\sqrt{2}$ has been chosen such that ϕ_k has L_2-norm equal to 1.

According to Fourier theory we can represent the solution of (5.2.1) with boundary condition (5.2.9) and initial condition (5.2.6) as

$$(5.2.10) \qquad u(t,x) = e^{rt} \sum_{k=1}^{\infty} a_k e^{-(k\pi)^2 t} \sin(k\pi x)$$

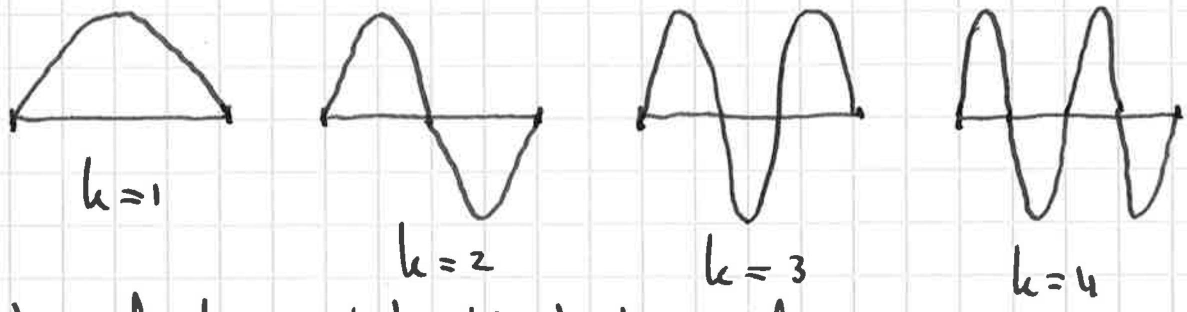
where the a_k are found from u_0 by

$$a_k = 2 \int_0^1 u_0(x) \sin(k\pi x)\, dx$$

[Extend u_0 to $[0,2]$ by $u_0(x) = -u_0(2-x)$, $1<x\leq 2$ and next extend u_0 periodically to $\mathbb{R}$]

We conclude that all solutions die out if $r<(k\pi)^2$ while if $r>(k\pi)^2$ any solution with $a_1 \neq 0$ grows exponentially. Note that $u_0 \geq 0$ with strict inequality on a set of positive measure implies $a_1>0$.

3.5



graphs of ϕ_k. Note that ϕ_1 is of one sign

Note that $\phi_k(1-x) = \phi_k(x)$ if k is odd

$\phi_k(1-x) = -\phi_k(x)$ if k is even

Note that the number of zero's in the open interval $(0,1)$ is equal to $k-1$

Exercise 5.2.1, 5.2.2, 5.2.3 Replace the boundary condition (5.2.9) by the no-flux (zero Neumann) boundary condition

$$(5.2.2) \quad \frac{\partial u}{\partial x}(t,0) = 0 = \frac{\partial u}{\partial x}(t,1)$$

and repeat the analysis. Show in particular that

$$\lambda_k = r - ((k-1)\pi)^2 \qquad k = 1, 2, 3, \dots$$

$$\phi_k(x) = \sqrt{2}\cos((k-1)\pi x) \qquad k = 2, 3, \dots$$

note: ϕ_1 does not change sign $\longleftrightarrow$ $\phi_1(x) = 1$

and that the solution is now given by

$$u(t,x) = c_1 e^{rt} + \sum_{m=1}^{\infty} b_m e^{\lambda_{m+1} t} \phi_{m+1}(x)$$

$$b_m = 2\int_0^1 u_0(x)\cos(m\pi x)\,dx \qquad m = 1, 2, \dots$$

and $$c_1 = \int_0^1 u_0(x)\,dx$$

Now $\phi_k(1-x) = \phi_k(x)$ if k is even

$\phi_k(1-x) = -\phi_k(x)$ if k is odd

and $k-1$ corresponds to the number of zero's of ϕ_k in $[0,1]$.

Exercise Replace the boundary condition now by the Robin (or "third kind") boundary condition

$$\frac{\partial u}{\partial x}(t,0) + \alpha_0 u(t,0) = 0 = \frac{\partial u}{\partial x}(t,1) + \alpha_1 u(t,1)$$

and repeat the analysis. [This is a bit more involved. The elaboration can be found in "Partial Differential Equations: an introduction" by W. A. Strauss, Wiley, 2008, Chapter 4, Section 4.3]

Important Observation The eigenvalues are real and they tend to $-\infty$ for $k \to \infty$. The eigenfunction ϕ_1 corresponding to the largest (:= principal or dominant) eigenvalue λ_1 is of one sign. And it is the only eigenfunction with that property.

We now briefly describe the general theory of elliptic equations that leads to these (and other) conclusions for a very large class of problems.

3.7/ Literature

H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2011

L.C. Evans, Partial Differential Equations, AMS, 2nd ed. 2010

D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 2nd ed., 2001

M. Chipot, Elliptic equations: an introductory course Birkhäuser, 2009

A. Henrot, Extremum problems for eigenvalues of elliptic operators, Birkhäuser, 2006

M. Renardy, R.C. Rogers, An introduction to partial differential equations, Springer, 1993

For linear problems, and in particular for spectral theory, it is convenient to work with $L_2(\Omega)$ as the underlying space, because this is a Hilbert space. For nonlinear problems, however, it is more convenient to work with $C(\bar{\Omega})$ as it makes the Nemitskii operator $(\hat{f}(\phi))(x) = f(\phi(x))$ well defined. Often the two approaches are combined by showing that a fractional power space or an interpolation space is continuously embedded into $C(\bar{\Omega})$ or even $C^1(\bar{\Omega})$, see

A. Friedman, Partial Differential Equations, 1976

D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer, 1981

A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, 1995

H. Amann, Linear and Quasilinear Parabolic Problems Birkhäuser, 1995

3.8/ We will not present this theory. A key point is that one gains regularity by solving an elliptic problem. As a consequence, the results of a spectral analysis are robust, i.e., they do not depend on the choice of the basic function space.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with a C^1 boundary $\partial\Omega$. Consider for a given $f \in L_2(\Omega)$ the Dirichlet problem

$$\text{(BVP)} \qquad \begin{aligned} L\phi &= f \quad &&\text{in } \Omega \\ \phi &= 0 &&\text{on } \partial\Omega \end{aligned}$$

where

$$L\phi = -\Delta\phi + c\phi$$

and c is a bounded measurable function on Ω with values in $\mathbb{R}$.

The Sobolev space $H^1(\Omega)$ consists of elements of $L_2(\Omega)$ that have weak first order partial derivatives (i.e., partial derivatives in the sense of distributions) that belong to $L_2(\Omega)$. Equipped with the inner product

$$\langle \phi, \psi \rangle_{H^1(\Omega)} = \int_\Omega \left(\phi\psi + \nabla\phi \cdot \nabla\psi \right)$$

$H^1(\Omega)$ is a Hilbert space.

On $H^1(\Omega)$ we introduce the continuous bilinear form

$$B[\phi, \psi] = \int_\Omega \left(\nabla\phi \cdot \nabla\psi + c\phi\psi \right)$$

Let $H_0^1(\Omega)$ denote the closure in $H^1(\Omega)$ of $C_c^\infty(\Omega)$, the space of C^∞ functions on Ω with compact support.

We say that $\phi \in H_0^1(\Omega)$ is a <u>weak solution</u> of (BVP) if

$$B[\phi,\psi] = (f,\psi)_{L_2(\Omega)} \qquad \forall\, \psi \in H_0^1(\Omega)$$

<u>Exercise</u> Use partial integration to motivate this definition

<u>Terminology</u> $B[\phi,\psi] = (f,\psi)_{L_2(\Omega)}$ $\forall\, \psi \in H_0^1(\Omega)$ is often called the <u>variational formulation</u> of (BVP), see page 3.18

<u>Theorem</u> (Lax–Milgram) Let H be a Hilbert space and let $B : H \times H \to \mathbb{R}$ be a bilinear mapping for which ~~such that~~ there exist constants C_1 and C_2 such that

$$|B[\phi,\psi]| \leq C_1 \|\phi\| \|\psi\| \quad , \qquad \phi,\psi \in H$$

(coercivity) $$C_2 \|\phi\|^2 \leq B[\phi,\phi] \quad , \qquad \phi \in H$$

Let $\tilde{f} : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique $\phi \in H$ such that

$$B[\phi,\psi] = \langle \tilde{f},\psi \rangle \qquad \forall\, \psi \in H$$

In order to apply this to the variational formulation of (BVP), we first of all observe that $\langle \tilde{f},\psi \rangle_{H_0^1(\Omega)} = (f,\psi)_{L_2(\Omega)}$ defines a bounded linear functional. Next we observe that the estimate for B involving the constant C_1 follows directly from the definition of B and the boundedness of the coefficient c in the differential operator L.

<u>Lemma</u> (Poincaré-Friedrichs)
There exists a constant $\gamma = \gamma(\Omega) > 0$ such that for every $\psi \in H_0^1(\Omega)$ we have

$$\int_\Omega |\psi|^2 \leq \gamma \int_\Omega |\nabla \psi|^2$$

<u>Exercise</u> Assume that $c \geq 0$. Derive the coercivity estimate with $C_2 = \frac{1}{\gamma + 1}$ with γ from the Poincaré-Friedrichs lemma

<u>Corollary</u> Let the constant μ be such that $c + \mu$ is nonnegative on Ω. Then there exists a unique weak solution $\phi \in H_0^1(\Omega)$ of the BVP

$$\begin{aligned} L\phi + \mu\phi &= f && \text{in } \Omega \\ \phi &= 0 && \text{on } \partial\Omega \end{aligned}$$

$\subset L_2(\Omega)$

<u>Definition</u> Define $G_\mu : L_2(\Omega) \to H_0^1(\Omega)$ by

$G_\mu(f) = \phi$, with ϕ the weak solution of the above BVP. The G refers to Green and we call G_μ the Green operator corresponding to μ

3.11 **Exercise** Show that the weak formulation of the eigenvalue problem

$$L\phi = \lambda\phi \quad \text{in } \Omega$$
$$\phi = 0 \quad \text{on } \partial\Omega$$

is equivalent to the eigenvalue problem

$$G_\mu(\phi) = \frac{1}{\lambda+\mu}\phi$$

Theorem (Rellich – Kondrachov; special case)
A bounded subset of $H_0^1(\Omega)$ is, considered as a subset of $L_2(\Omega)$, relatively compact (meaning that its closure is compact)

Before formulating some important conclusions, we make a few useful observations

i) B is symmetric : $B[\phi,\psi] = B[\psi,\phi]$

ii) the map $G_\mu : L_2(\Omega) \to H_0^1(\Omega)$ is linear and bounded : let $\phi = G_\mu(f)$ then

$$C_2 \|\phi\|^2_{H_0^1} \leq B[\phi,\phi] = (f,\phi)_{L_2} \leq \|f\|_{L_2}\|\phi\|_{L_2} \leq \|f\|_{L_2}\|\phi\|_{H_0^1}$$

iii) since $H_0^1(\Omega) \subset L_2(\Omega)$ we may also view G_μ as a bounded linear map from $L_2(\Omega)$ into $L_2(\Omega)$

Theorem The bounded linear operator $G_\mu : L_2(\Omega) \to L_2(\Omega)$ is
a) compact (i.e., maps bounded sets into relatively compact sets)
b) self-adjoint (i.e., $(G_\mu f, g) = (f, G_\mu g)$)
c) positive (i.e., $(G_\mu f, f) > 0$ if $f \neq 0$)

3.12 <u>Proof</u> a) A bounded subset of $L_2(\Omega)$ is mapped to a bounded subset of $H_0^1(\Omega)$, see observation ii above, so to a relatively compact subset of $L_2(\Omega)$ (Rellich–Kondrachov)

b) The symmetry of B allows us to write

$$(f, \mathcal{G}_\mu g)_{L_2} = B[\mathcal{G}_\mu f, \mathcal{G}_\mu g] = B[\mathcal{G}_\mu g, \mathcal{G}_\mu f]$$
$$= (g, \mathcal{G}_\mu f)_{L_2}$$

where the first and the last step use that

$B[\mathcal{G}_\mu h, \psi] = (h, \psi)_{L_2}$ for all $\psi \in H_0^1(\Omega)$, so certainly for ψ of the form $\mathcal{G}_\mu g$ or $\mathcal{G}_\mu f$

c) Similarly we have

$$(\mathcal{G}_\mu f, f)_{L_2} = (f, \mathcal{G}_\mu f)_{L_2} = B[\mathcal{G}_\mu f, \mathcal{G}_\mu f]$$
$$\geq C_2 \| \mathcal{G}_\mu f \|_{H_0^1}$$

which is positive and only equal to zero if $\mathcal{G}_\mu f = 0$

But when $\mathcal{G}_\mu f = 0$ we have that $(f, \psi)_{L_2} = B[\mathcal{G}_\mu f, \psi] = 0$ for all $\psi \in H_0^1(\Omega)$ and therefore (since $H_0^1(\Omega)$ is dense in $L_2(\Omega)$) that $f = 0$ $\square$

Corollary The spectrum $\sigma(\mathcal{G}_\mu)$ consists of 0 and a sequence of positive numbers Λ_i, ordered such that $\Lambda_{i+1} \leq \Lambda_i$ and converging to 0 for $i \to \infty$, which are eigenvalues. So to Λ_i corresponds $\phi_i \in L_2(\Omega)$ with $\|\phi_i\|_{L_2} = 1$ such that

$$\mathcal{G}_\mu \phi_i = \Lambda_i \phi_i$$

The $\{\phi_i\}_{i=1}^{\infty}$ form a complete orthonormal system in $L_2(\Omega)$, so in particular for any $\psi \in L_2(\Omega)$ we have

$$\lim_{N\to\infty} \Big\| \sum_{i=1}^{N} (\phi_i, \psi)_{L_2} \phi_i - \psi \Big\|_{L_2} = 0$$

The equation $\mathcal{G}_\mu \phi - \Lambda_i \phi = f$ has a solution if and only if $(\phi_j, f)_{L_2} = 0$ for all j such that $\Lambda_j = \Lambda_i$

Remarks i) This follows from the spectral theory for compact operators, with self-adjointness and positivity giving some extra information

ii) Note that the ϕ_i belong to $H_0^1(\Omega)$

iii) The last part relates to the Fredholm alternative (saying that for $T: H \to H$ compact either $\mathcal{R}(I-T) = H$ or $\mathcal{N}(I-T) \neq \{0\}$), more precisely to $\mathcal{R}(I-T) = \mathcal{N}(I-T^*)^{\perp}$ which simplifies to $\mathcal{R}(I-T) = \mathcal{N}(I-T)^{\perp}$ if T is self-adjoint.

iv) From the Exercise at the top of page 3.11 we infer that L has, in the weak sense, eigenvector ϕ_i with corresponding eigenvalue $\lambda_i = -\mu + \frac{1}{\Lambda_i}$ converging to $+\infty$ for $i \to \infty$

We call λ_1 the <u>principal eigenvalue</u> of L and ϕ_1 the <u>principal eigenfunction</u>. In the context of the parabolic problem $\frac{\partial u}{\partial t} + Lu = 0$ it is more natural to look at the eigenvalues $\mu - \frac{1}{\lambda_i}$ of $-L$. These converge to $-\infty$ for $i \to \infty$. The eigenvalue $\mu - \frac{1}{\lambda_1}$ is the rightmost (in $\mathbb{C}$) eigenvalue and its sign governs the asymptotic behaviour. In this context we often speak about the <u>dominant eigenvalue</u>.

<u>Theorem</u>

(i) variational characterization $\lambda_1 = \min\limits_{\substack{\psi \in H_0^1(\Omega) \\ \psi \neq 0}} \dfrac{B[\psi,\psi]}{\|\psi\|_{L_2}^2}$

(Rayleigh quotient)

The minimum is attained for $\psi = \phi_1$

(ii) ϕ_1 is of one sign in Ω

(iii) λ_1 is simple, i.e., if $L\psi = \lambda_1 \psi$ in Ω, $\psi = 0$ on $\partial\Omega$, then ψ is a multiple of ϕ_1

(iv) if $L\psi = \lambda\psi$ in Ω, $\psi = 0$ on $\partial\Omega$ and ψ is of one sign in Ω then $\lambda = \lambda_1$

<u>Remarks</u> 1. If we consider $L\phi = -\Delta\phi$ (so take c identically zero) then λ_1 yields ~~that~~ $\gamma = \frac{1}{\lambda_1}$ as the best possible constant in the Poincaré-Friedrichs estimate of page 3.10

3.15

2. If $L\phi = -\Delta\phi$ (so again we take $c=0$), then we can vary the domain Ω. So, writing $\lambda_1 = \lambda_1(\Omega)$ to denote the principal eigenvalue of $-\Delta$ on the domain Ω, we conclude that
$$\lambda_1(\Omega) \geq \lambda_1(\tilde{\Omega}) \quad \text{if } \Omega \subset \tilde{\Omega}$$
(use the variational characterization and the fact that an element of $H_0^1(\Omega)$ yields, when defined as zero in $\tilde{\Omega} \setminus \Omega$, an element of $H_0^1(\tilde{\Omega})$ with the same Rayleigh quotient)

3. One way of proving that ϕ_1 is of one sign makes use of the Krein-Rutman Theorem, which is an infinite dimensional variant of the Perron-Frobenius Theorem:

<u>Theorem</u> (Krein-Rutman) Let X be a Banach space and $K \subset X$ a cone (i.e., a closed convex set such that $\lambda K \subset K$, $\forall \lambda \geq 0$ and $K \cap (-K) = \{0\}$) such that $K - K = \{\phi - \psi : \phi, \psi \in K\}$ is dense in X. Let $T: X \to X$ be a <u>compact</u> linear operator which is <u>positive</u> in the sense that $T(K) \subset K$. Let $r(T)$ be the <u>spectral radius</u> (so $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$). ~~Then~~ Assume that $r(T) > 0$. Then $r(T)$ is an eigenvalue of T and the corresponding eigenvector can be chosen such that it belongs to $K \setminus \{0\}$.

If K has non-empty interior and T maps $K \setminus \{0\}$ into the interior of K, then $r(T) > 0$ holds and, moreover, $r(T)$ is a <u>simple</u> eigenvalue of T and the corresponding eigenvector in K belongs to the interior of K. Finally $|\lambda| < r(T)$

for all eigenvalues λ of T other than $r(T)$.

Note that in order to apply the second part of this theorem, we need to work in $C(\bar{\Omega})$, since the positive cone in $L_2(\Omega)$ has empty interior.
Also note that we need the <u>maximum principle</u> (to be treated soon) to show positivity, i.e., to show that the relevant operator T maps the cone K consisting of positive functions into itself.

Before turning to the Neumann problem with no-flux boundary conditions, we make some remarks about the regularity of solutions. Let ϕ be a weak solution of (BVP) in the case of $\Omega \subset \mathbb{R}$, i.e., one-dimensional space variable. Writing the variational formulation as

$$\int \phi' \psi' = \int (f - c\phi)\psi \qquad \forall\, \psi \in H_0^1(\Omega)$$

we see right away that $\phi' \in H^1(\Omega)$, hence $\phi \in H^2(\Omega)$.
So we gain <u>two derivatives</u> when going from the given f to the corresponding solution. If f, c (and in the higher dimensional case also Ω) are smooth, we can look at the differentiated version of (BVP) and deduce that once again ~~that~~ ϕ has a degree of differentiability that exceeds that of f by two.

Now consider

$$(\mathrm{BVP})_{\text{no-flux}} \quad \begin{aligned} L\phi &= f && \text{in } \Omega \\ \frac{\partial \phi}{\partial n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

where as before

$$L\phi = -\Delta\phi + c\phi \qquad \text{and } f \in L_2(\Omega) \text{ is given}$$

3.17/ The remarkable fact is that a weak solution of $(BVP)_{\text{no-flux}}$ is provided by $\phi \in H^1(\Omega)$ ~~set~~ such that

$$B[\phi, \psi] = (f, \psi)_{L_2(\Omega)} \qquad \forall\, \psi \in H^1(\Omega)$$

even though, strictly speaking, $\frac{\partial \phi}{\partial n}\Big|_{\partial\Omega}$ is not defined if we only know that $\phi \in H^1(\Omega)$. So here it is essential to elaborate the regularity results after establishing, by Lax-Milgram, that the variational formulation yields a unique candidate solution. After showing that that solution belongs to $H^2(\Omega)$, one can check that the no-flux boundary condition is indeed satisfied in an appropriate sense.

As in the case of the Dirichlet problem with zero boundary conditions, we find

- there exists a countable set of eigenvalues of L with no-flux boundary conditions, converging to $+\infty$
- the corresponding eigenvectors form a complete orthonormal system in $L_2(\Omega)$
- the Fredholm theory applies
- the eigenfunction corresponding to the principal eigenvalue is of one sign while (linear combinations of) all other eigenfunctions change sign
- the principal eigenvalue is simple and equals

$$\min_{\substack{\psi \in H^1(\Omega) \\ \psi \neq 0}} \frac{B[\psi,\psi]}{\|\psi\|^2_{L_2}}$$ (the minimum is attained by the corresponding eigenfunction)

- hence this principal eigenvalue is <u>smaller</u> than the one corresponding to zero Dirichlet b.c. (simply because $H_0^1(\Omega) \subset H^1(\Omega)$)

Explanation of the term "variational formulation" introduced on page 3.9

$H_0^1(\Omega) \to \mathbb{R}$

Consider the functional $\phi \longmapsto \frac{1}{2}B[\phi,\phi] - (f,\phi)$

Since

$$\tfrac{1}{2}B[\phi+\varepsilon\psi, \phi+\varepsilon\psi] - (f,\phi+\varepsilon\psi) - \tfrac{1}{2}B[\phi,\phi] + \varepsilon(f,\phi)$$

$$= \tfrac{1}{2}2\varepsilon B[\phi,\psi] - \varepsilon(f,\psi) + \varepsilon^2 B[\psi,\psi]$$

we see right away that ϕ is a critical point of this functional iff $B[\phi,\psi] - (f,\psi) = 0$, $\forall \psi \in H_0^1(\Omega)$

Since the functional is bounded below and coercive, it must have a minimum. That the solution of the Dirichlet problem is found by minimizing a functional is often called "Dirichlet's Principle".

This ends our excursion into the theory of elliptic partial differential equations. We now know that (5.2.8) and (5.2.10) generalize to

$$u(t,x) = \sum_{k=1}^{\infty} d_k e^{\lambda_k t} \phi_k(x)$$

$$= d_1 e^{\lambda_1 t} \phi_1(x) + o(e^{\lambda_1 t}), \quad t \to \infty$$

where ϕ_1 is a positive function and $d_1 = (\phi_1, u_0) > 0$ whenever $u_0 \geq 0$, $u_0 \neq 0$. Hence the sign of λ_1 is of crucial importance for the asymptotic behaviour for large time. Beware that now, as explained at the top of page 3.14, the λ_i are the eigenvalues of $-L$, so they converge to $-\infty$ for $i \to \infty$.

Exercise Recall the scaling at the bottom of page 3.2
Consider the problem (5.2.1), (5.2.9), see page 3.2, with $r>0$.
Explain the notion of "critical patch size" by showing that $\lambda_1 < 0$ for small L but $\lambda_1 > 0$ for large L.

Remarks (i) You may remember from Fourier theory that the decay of the coefficients d_k with k, reflects the smoothness of the function that is represented by the series. Now note that when $t>0$ the coefficients $d_k e^{\lambda_k t}$ are "damped" by the factor $e^{\lambda_k t}$ and that $\lambda_k \to -\infty$ for $k \to \infty$. This shows the smoothing effect of the diffusion equation.

(ii) We have $u_0 = \sum_{k=1}^{\infty} d_k \phi_k$ in $L_2(\Omega)$ but, as you may also remember from Fourier theory, convergenge in $C(\bar{\Omega})$ is a subtle matter. In fact, if we choose for (5.2.1)-(5.2.9) the function u_0 in (5.2.6) such that it is non-zero in the boundary points, we cannot have that $u(t,x) - u_0(x) \to 0$ as $t \downarrow 0$ uniformly on $\bar{\Omega}$, since $u(t,\cdot)$ is zero in the boundary points for all $t>0$. (In terms of semigroups: we have $\lim_{t\downarrow 0} T(t)\phi = \phi$ for $\phi \in C_0(\Omega)$ but not for $\phi \in C(\bar{\Omega})$ that are not zero at the boundary.)

(iii) If $(L\phi)(x) = -\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}(x)\right) + q(x)\phi(x)$ with $p \in C^1[0,1]$ and $q \in C[0,1]$ we usually call the eigenvalue problem a Sturm-Liouville problem. There exists an extensive theory for such problems, see Brézis' book and the references given there (or see G. Teschl, Ordinary Differential Equations and Dynamical Systems, AMS, 2012, Ch. 5)

(iv) See Exercise 5.2.6 for a two-dimensional domain where, due to symmetry, we can in some sense reduce the eigenvalue problem to two one-dimensional problems. This helps to pinpoint the nodal lines that separate nodal domains (i.e., subdomains on which the function doesn't change sign). Not much seems to be known in general about these nodal domains (see Section 1.3.3 in the book of Henrot listed on page 3.7 for a few remarks)

Finally, let us look at the linear <u>system</u> of equations

$$\frac{\partial u}{\partial t} = d\,\Delta u + M u$$

where $\Omega \subset \mathbb{R}^n$, $u(t,x) \in \mathbb{R}^k$, $M \in \mathbb{R}^{k\times k}$ and

$$d = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_k \end{pmatrix}$$

is a $k \times k$ diagonal matrix with $d_j > 0$, $j = 1, \dots, k$

Separation of variables now involves <u>three</u> factors

$$u(t,x) = e^{\lambda t}\,\phi(x)\,\psi \quad \text{with } \phi(x) \in \mathbb{R},\ \psi \in \mathbb{R}^k$$

Substituting this into the equation we find

$$\lambda\,\phi(x)\psi = (\Delta\phi)(x)\,d\psi + \phi(x)\,M\psi$$

So if $\phi(x) \neq 0$ we necessarily should have $\Delta\phi = \mu\phi$

3.21/ and so, provided all components of u are subject to the same boundary condition, we let μ, ϕ be such that the problem $\Delta\phi = \mu\phi$ + boundary condition has a nontrivial solution. The theory developed above applies to this problem for μ, ϕ. In particular we find countably many μ_i with associated ϕ_i. For each μ_i we subsequently consider the $\mathbb{R}^k$ eigenvalue problem

$$(M + \mu_i d)\,\psi = \lambda\psi$$

So for each μ_i we have eigenvalues $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik}$

Note that

- the order of the μ_i is not necessarily reflected in the corresponding collection of λ's

(more precisely, it may happen that $\mu_{i+1} < \mu_i$ and yet for some combination of $j, l \in \{1, \ldots, k\}$ we have that

$$\lambda_{(i+1)j} > \lambda_{il}$$
)

- it is quite conceivable that the eigenvector ψ corresponding to the λ with the largest real part has both positive and negative components

Note Section 5.3 serves to make you aware that on unbounded domains the growth away from zero may not be exponential, but instead involve wave like expansion with a characteristic speed. So compactness is an essential component of the theory described in this lecture.