Section 3 The hair trigger effect and the asymptotic speed of propagation

We now consider the initial value problem for (11.1) with f as described below that equation. More precisely we complement (11.1) by

$$u(0,x) = u_0(x) \qquad \text{with} \quad 0 \le u_0(x) \le 1$$

Theorem 9 (hair trigger effect)

If $u_0(x)$ is not identically zero, $\lim_{t\to\infty} u(t,x) = 1$, $-\infty < x < \infty$

Proof By the comparison Theorem 5 of page 9.25, $u(h,x) > 0$ if $h > 0$. Since $f'(0) > 0$ we have that

$$f(q) \geq \tfrac{1}{2} f'(0) q$$

for q positive and sufficiently small, say $0 \le q \le \bar{q}$.

Define

$$q^\varepsilon(x) = \varepsilon \sin\left(\sqrt{\tfrac{1}{2} f'(0)}\, x\right)$$

then

$$\frac{d^2}{dx^2} q^\varepsilon(x) + f(q^\varepsilon(x)) \geq 0 \quad \text{for } 0 \le \varepsilon \le \bar{q}, \quad 0 \le x \le \frac{\pi}{\sqrt{\frac{1}{2} f'(0)}}$$

Now choose ε so small that

$$u(h,x) \geq q^\varepsilon(x) \qquad \text{for } 0 \le x \le \frac{\pi}{\sqrt{\frac{1}{2} f'(0)}}$$

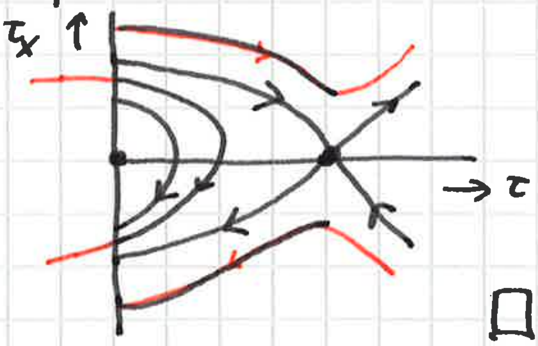
[max of subsolutions is a subsolution ← we use this without giving the proof]

then, by the same comparison theorem, $u(t+h,x)$ exceeds the solution starting at time zero with initial condition q^ε. Since q^ε is a subsolution, the latter solution is nondecreasing, so has a limit. If we can show that this limit is the function that is identically equal to one, we are done (since $\bar{\varphi} \equiv 1$ is a supersolution

11.15/ and $\Psi \equiv 0$ is a subsolution, all solutions starting with values between 0 and 1 remain bounded below by 0 and above by 1 for all time).
In Lemma 10 below it is shown that the solution with initial condition $\cancel{\tau(x) \equiv 0}$ q^ε converges to $\tau(x)$ where $\tau(x)$ is the smallest solution of

$$\tau_{xx} + f(\tau) = 0 \qquad -\infty < x < \infty$$

~~wit~~ which satisfies $0 \leq \tau(x) \leq 1$ as well as $\tau(x) \geq q^\varepsilon(x)$ for $x \in \left(0, \frac{\pi}{\sqrt{\frac{1}{2} f'(0)}}\right)$. The phase plane picture



(recall page 5.4) shows at once that these characteristics of τ imply that τ is identically equal to 1 $\square$

Lemma 10 Let q with $0 \leq q(x) \leq 1$ satisfy

$$q'' + cq' + f(q) \geq 0 \text{ in } (a,b)$$

where $-\infty \leq a < b \leq \infty$. If $a > -\infty$ assume that $q(a) = 0$ and if $b < \infty$ assume that $q(b) = 0$. Let $u(t,x)$ denote the solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} + f(u)$$

satisfying

$$u(0,x) = \begin{cases} q(x) & a < x < b \\ 0 & x \in \mathbb{R} \setminus (a,b) \end{cases}$$

Then $u(t,x)$ is, for each x, a nondecreasing function of t and

$$\lim_{t\to\infty} u(t,x) = \tau(x)$$

uniformly on bounded x-intervals, where $\tau(x)$ is the smallest solution of

$$\tau'' + c\tau' + f(\tau) = 0, \quad -\infty < x < \infty,$$

which satisfies $0 \le \tau(x) \le 1$ as well as $\tau(x) \geqslant q(x)$, $a < x < b$.

Proof The monotonicity follows, as before, from repeated application of the comparison principle. Since any solution of $\varphi'' + c\varphi' + f(\varphi) = 0$ that exceeds $u(0,x)$ is a supersolution, the limit τ is certainly bounded from above by the smallest such φ. So it only remains to prove that the limit τ is a steady state, i.e., satisfies the steady state equation. This concerns a variant of Lemma 9 on page 9.29, but since we now work on all of $\mathbb{R}$, the details are somewhat different. One option is to argue as in the alternative proof of Lemma 4 sketched on page 11.9, i.e., to use compactness, see pages 102–103 of nonlineardiffusion@wisyllabus.pdf Chapter IV. $\square$

Lemma 10 in fact allows us to make the assertion of Theorem 9 slightly stronger:

$$u(t,x) \to 1 \text{ for } t\to\infty \text{ uniformly on compact sets}$$

The results of Section 2 on travelling waves make it tempting to conjecture that for initial conditions u_0 with compact support

$$\forall c > c_0 \quad \lim_{t\to\infty} \sup\{u(t,x) : |x| \geq ct\} = 0$$

$$\forall c < c_0 \quad \lim_{t\to\infty} \min\{u(t,x) : |x| \leq ct\} = 1$$

and to call c_0 the asymptotic speed of propagation to reflect these properties. This notion was introduced (and developed for one-dimensional space) by D.G. Aronson and H.F. Weinberger in a contribution to Springer Lecture Notes in Math. 446 published in 1975 and then fully developed in their beautiful classic paper

Multidimensional nonlinear diffusion arising in population genetics, Advances in Mathematics (1978) 30 : 33–76

(recall Section 5.3 of syllabus_chs.pdf and see amsbookreview.pdf in the Bibliography for a discussion in the context of models describing the spatial spread of infectious diseases)

For $\mathbb{R}$ the set $|x| \geq ct$ naturally splits into the two subsets $x \geq ct$ and $x \leq -ct$. Therefore we shall formulate and prove a one-sided version of the limsup $= 0$ result. As an immediate consequece of the comparison principle we have

Lemma 11 Let $c > c_0$ hold. Denote by w_c the travelling wave solution with this speed and normalized by $w_c(0) = \frac{1}{2}$. Then $u_0(x) \leq w_c(x + x_0) \Rightarrow u(t,x) \leq w_c(x + x_0 - ct)$

11.18 The idea is of course to exploit the freedom in choosing x_0 to achieve that $u_0(x) \leq W_c(x+x_0)$. If $u_0(x)$ has an upper bounded support and is bounded above by $1-\varepsilon$, $\varepsilon>0$, this can indeed be done by taking $|x_0|$ large and x_0 negative. If $u_0(x)$ has compact support and is not bounded away from the value 1, we wait a little while and then there is a gap between $\max\{u(t,\cdot)\}$ and 1 and we start from this $u(t,\cdot)$ rather than from the true initial condition.

<u>Theorem 12</u> Let u_0 have compact support and be bounded away from the value 1. Let $c>c_0$. Then

$$\lim_{t\to\infty} \sup\{u(t,x) : x \geq ct\} = 0$$

<u>Proof</u> Choose $\tilde{c}$ with $c>\tilde{c}>c_0$. By appropriate choice of x_0 we have, with the notation of Lemma 11

$$u_0(x) \leq W_{\tilde{c}}(x+x_0)$$

and therefore

$$u(t,x) \leq W_{\tilde{c}}(x+x_0-\tilde{c}t)$$

So for $x \geq ct$ we have, since $W_{\tilde{c}}$ is monotonically decreasing

$$u(t,x) \leq W_{\tilde{c}}(x_0+(c-\tilde{c})t)$$

and since $c-\tilde{c}>0$ the right hand side tends to $W_{\tilde{c}}(\infty)=0$ for $t\to\infty$. $\square$

In conclusion of this section we formulate and prove a remarkable variant of the Hair-Trigger Theorem 9.

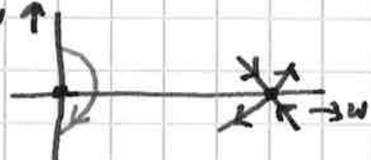
<u>Theorem 13</u> If u_0 is not identically equal to zero, then for any $c \in (0, c_0)$ and any $x \in (-\infty, \infty)$ we have

$$\lim_{t\to\infty} u(t, x+ct) = 1$$

<u>Proof</u> Let $c \in (0, c_0)$. Define $\bar{u}(t,x) = u(t, x+ct)$ then

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial x^2} + c\frac{\partial \bar{u}}{\partial x} + f(\bar{u}) \qquad (11.14)$$

Our first step is to show that for the system $\begin{aligned} w' &= v \\ v' &= -cv - f(w) \end{aligned}$ there is a piece of orbit that connects the positive v-axis to the negative v-axis via the $w>0$ halfplane. If $c_0 = 2f'(0)$ this follows right away from the fact that $(0,0)$ is a stable spiral point.

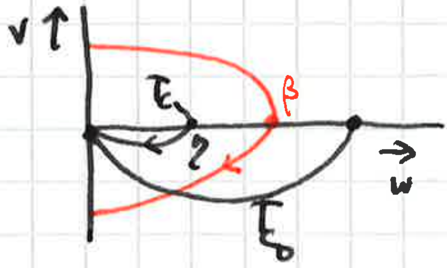


So assume $c_0 > 2\sqrt{f'(0)}$. While discussing Example 8 we noted that among the orbits approaching the stable node $(0,0)$ there is, if we restrict to $w>0$, exactly one that is tangent to $\binom{1}{\lambda_-}$. Call this orbit T_c. We also noted that c_0 is characterized by T_{c_0} coinciding with the unstable manifold of $(1,0)$. We claim that for $c<c_0$ the orbit T_c intersects the w-axis in some point, say $(\eta, 0)$. To substantiate this claim, we note

i) $$2\frac{d\lambda_-}{dc}\Big|_{(0,0)} = -1 - \frac{c}{\sqrt{c^2 - f'(0)}} < 0$$

ii) along T_{c_0} the vectorfield $(v, -cv - f(w))$ points <u>out</u> of the region bounded by T_{c_0} and the w-axis (recall page 11.5)

11.20/ Point i) tells us that T_c lies above T_{c_0}. If T_c would intersect T_{c_0}, the orbit would forward in time enter the region between T_{c_0} and the w-axis.



According to Point ii), this is impossible. So T_c must intersect the w-axis and we call the corresponding w-value η. Choose $\beta \in (\eta, 1)$. Consider the orbit through $(\beta, 0)$. Forward in time this orbit has $v < 0$, so w decreases. As v remains bounded, the orbit either intersects the v-axis or it converges to $(0,0)$. Since $\lambda_- < \lambda_+ < 0$, the T_c orbit "shields" $(0,0)$ as a target for the orbit through $(\beta, 0)$ from the quarter plane $w > 0, v < 0$. So the orbit must intersect the v-axis before, possibly, converging to $(0,0)$. If we follow the orbit through $(\beta,0)$ <u>backward</u> in time, we find again that w decreases and that v remains bounded. Since neither $\binom{1}{\lambda_-}$ nor $\binom{1}{\lambda_+}$ points into $\{(w,v) : w>0, v>0\}$, convergence to $(0,0)$ is impossible. So again we find that the orbit must intersect the v-axis.

Our second step is to associate a function $\tilde{q}_c(x)$ with the orbit through $(\beta, 0)$. If we let $x = a$ correspond to the intersection with the positive v-axis and $x = b$ with the intersection with the negative v-axis, we are still free to choose a (because of translation invariance), but $b-a$ is a fixed quantity. Note that $\tilde{q}_c$ corresponds to the w-component and that accordingly $\tilde{q}_c(a) = 0 = \tilde{q}_c(b)$.

The third step is to apply Lemma 10 with $\tilde{q}_c$ in the role of q. The phase plane information tells us that the resulting limit τ is identically equal to 1 (recall that $(0,1)$ is a saddle point and that orbits staying to the left from its stable - and unstable manifold in the region $w<1$ must intersect the v-axis, so become negative if we consider the w-component).

Our fourth step is to show that we can use the solution of (11.14), with the initial condition as in Lemma 10 with $q = \tilde{q}_c$, as a lower solution. So in particular we want to show that there exists $\bar{t}$ such that

$$\bar{u}(\bar{t},x) \geq \tilde{q}_c(x) \qquad , \qquad a \leq x \leq b$$

By Theorem 9 we know that for given $b-a$ and given $\gamma \in (\beta,1)$ there exists a time $\bar{t}$ such that

$$u(\bar{t},x) > \gamma \qquad \text{for} \quad 0 \leq x \leq b-a$$

Now choose $a = -c\bar{t}$ and $b = -c\bar{t} + b - a$ (which, weird as it may look, makes sense since we consider $b-a$ as the given constant; yet, I admit, ~~that~~ this is bad notation) Then

$$\bar{u}(\bar{t},x) = u(\bar{t}, x + c\bar{t}) > \gamma \quad \text{for} \quad a \leq x \leq b$$

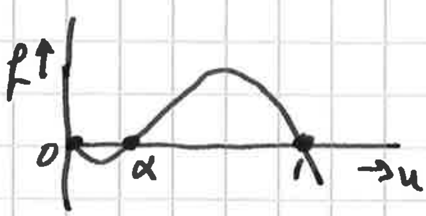
The final fifth step is to invoke the comparison Theorem 5 of page 9.25 and conclude that

$$\lim_{t\to\infty} u(t,x+ct) = \lim_{t\to\infty} \bar{u}(t,x) = 1 \qquad \square$$

In essentially the same manner, but by using slightly more complicated auxiliary functions, Aronson and Weinberger prove

$$\forall c < c_0 \qquad \lim_{t\to\infty} \min\{u(t,x) : |x| \leq ct\} = 1$$

11.22 **Section 4 The bistable case**



Now assume that the graph of f looks like the adjacent picture. More precisely, assume for some $\alpha \in (0,1)$ we have that f is zero for $u = 0, \alpha, 1$, $f < 0$ on $(0,\alpha)$, $f > 0$ on $(\alpha,1)$, $f'(0) < 0$, $f'(\alpha) > 0$, $f'(1) < 0$. The results of Section 2 tell us that waves connecting α and 1 exist for $c \geq c_0^{(\alpha,1)} \geq 2\sqrt{f'(\alpha)}$ while waves connecting α and 0 exist for $c \leq c_0^{(0,\alpha)} \leq -2\sqrt{f'(\alpha)}$. But does a wave connecting 0 and 1 exist? And if so, which way does it travel (given that it is a decreasing function with limit 1 at $-\infty$ and limit 0 at $+\infty$; note that the corresponding increasing function obtained by reflection travels in the other direction).

Lemma 14 If a decreasing profile connecting 1 and 0 exists and c denotes its speed, then

$$\operatorname{sign} c = \operatorname{sign} \int_0^1 f(w)\,dw$$

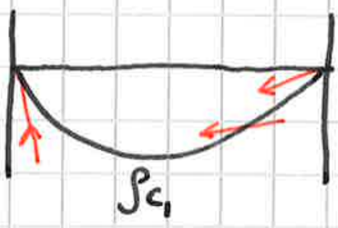
Proof We multiply the equation $w'' + cw' + f(w) = 0$ by w' and next integrate over $(-\infty,\infty)$. Since $\lim_{\xi \to \pm\infty} w'(\xi) = \lim_{\xi \to \pm\infty} v(\xi) = 0$, the first term does not contribute and we obtain the identity

$$c \int_{-\infty}^{\infty} (w')^2 + \int_{-\infty}^{\infty} w' f(w) = 0$$

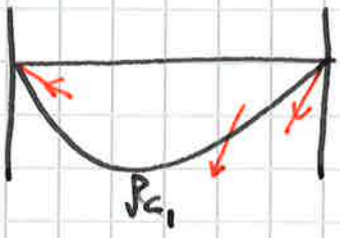
11.23/ Now note that $\int_{-\infty}^{\infty} w'(\xi) f(w(\xi))\, d\xi = \int_1^0 f(w)\, dw$ $\square$

Lemma 15 There is at most one speed

Proof Suppose a $(1,0) \to (0,0)$ heteroclinic connection exists for speed c_1. Let $v = \rho(w)$ be the corresponding curve in $\{(w,v) : 0 \leq w \leq 1,\ v \leq 0\}$. Exactly as in the proof of Lemma 1 (presented before the statement of the lemma itself), one proves that the vector field for $c \neq c_1$ either points inward the region bounded by ρ and the w-axis or outward, depending on the sign of $c - c_1$. The unstable manifold at $(1,0)$ and the stable manifold at $(0,0)$ "turn" accordingly if c is moved away from c_1 (just differentiate (11.6) with respect to c and use (11.7)).



$c > c_1$

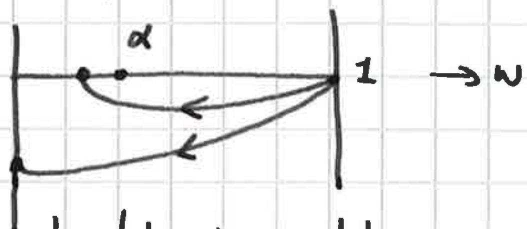


$c < c_1$

It follows that if the unstable manifold ~~be~~ of $(1,0)$ "starts" inside, it cannot get outside while if it starts outside it cannot get inside, so it cannot "do" what is needed to connect to the stable manifold of $(0,0)$. $\square$

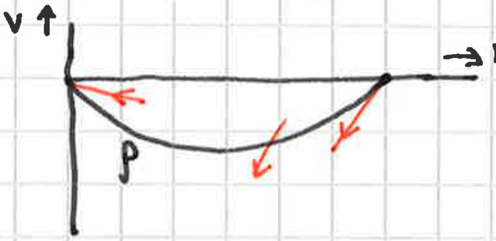
11.24 <u>Theorem 16</u> There exists a unique speed

"<u>Proof</u>" Essentially we use an intermediate value argument. For $c = c_0^{(\alpha,1)}$ the unstable manifold of $(1,0)$ "ends" at $(\alpha,0)$ and for slightly smaller values of c it will intersect the w-axis in a point with $0<w<\alpha$. As we shall show below, for c a lot smaller it intersects the v-axis in a point with $v<0$. A continuity argument (that, as already noted in the "Proof" of Lemma 4, is not easily made precise) establishes that it should reach $(0,0)$ for an intermediate value.



To show that for suitable c it intersects the negative v-axis, we choose a curve $v = \rho(w)$, with $\rho(0) = 0 = \rho(1)$, $\rho'(0) < 0$, $\rho'(1) > 0$, $\rho(w) < 0$ for $0<w<1$ and verify that for

$$c < \inf_{0<w<1} \left\{ -\rho'(w) - \frac{f(w)}{\rho(w)} \right\}$$

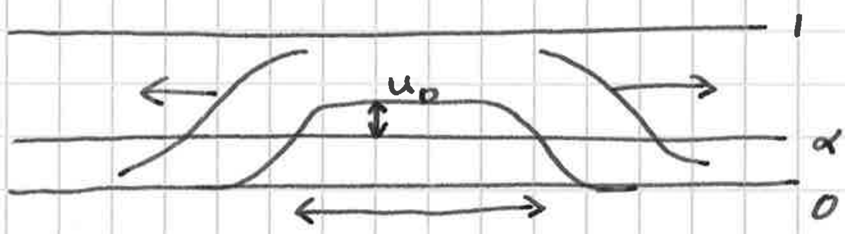


the vector field points out, the unstable manifold of $(1,0)$ "starts" outside, and the stable manifold of $(0,0)$ approaches $(0,0)$ from inside. So the stable manifold cannot go to $(0,0)$. As it also remains bounded in the strip $0 \leq w \leq 1$, it has to intersect the negative v-axis. $\square$

Next consider the initial value problem. As a rule, solutions either converge to $u \equiv 0$ or to $u \equiv 1$. They can do so uniformly in x (for instance, if $0 \leq u_0(x) \leq \alpha - \varepsilon$ for $-\infty < x < \infty$) but also non-uniformly.

11.25 See Y. Du, H. Matano, Convergence and sharp thresholds for propagation in nonlinear diffusion problems, J. Eur. Math. Soc. (2010) 12 : 279–312 for recent results as well as references to older literature.

If we introduce a perturbation $u_0(x)$ of the extinction state $u \equiv 0$, it should be sufficiently above the threshold value α over a sufficiently large interval



in order for convergence to $u \equiv 1$ to happen, and then the convergence is typically in the form of two travelling waves, one travelling to the right and the other to the left. The stability of the travelling wave was established by Fife and McLeod.

So also in the bistable case we find that a transition takes place with a well-defined speed, but now the perturbation needs to exceed a threshold if it is to trigger a transition.