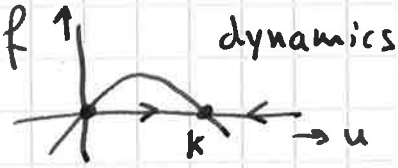
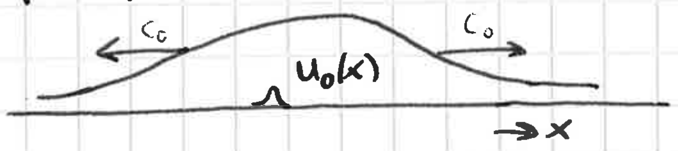
1.1/

$$\begin{cases} \dot{u} = f(u) \\ u(0) = u_0 \end{cases}$$



Now $u = u(t,x)$



$$\begin{cases} \dfrac{\partial u}{\partial t} = d\dfrac{\partial^2 u}{\partial x^2} + f(u) \\ u(0,x) = u_0(x) \end{cases}$$

Hair Trigger Effect

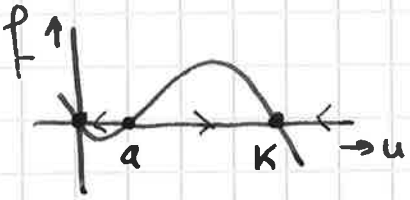
$\lim_{t \to \infty} u(t,x) = K$ unif. on compact x-sets

1937 Fisher – KPP (Kolmogorov, Petrovskii, Piskunov)

1975, 1978 Aronson & Weinberger Asymptotic Speed of Propagation

$$c_0 = 2\sqrt{d f'(0)}$$

Finite domain



but for system: pattern formation by Turing instability (diffusion driven instability)

---

1. The dynamical systems perspective
2. On individuals and populations
3. Derivation of the (linear) diffusion equation

The state of an individual at time t should summarize all information about the past (i.e., times prior to t) that is relevant for predicting the future of that individual

e.g. x = position in space

1.2/ contrast with particle in Newtonian/classical mechanics: also velocity matters

Also note that a photo would reveal position, but not velocity and that sometimes only observable directly quantities are called state variables. Perhaps that underlies the use of "phase space" for what we call "state space", i.e., the set of all possible states, e.g. $\boxed{\mathbb{R} =: \Omega}$

If, in principle, we can determine the state at a later time with certainty on the basis of knowledge of the current state, we deal with a deterministic dynamical system. More generally there is an element of chance involved and we can only determine the probability distribution of the state at some specified time into the future. So conceptually we deal with

(Feller) Transition Function

(2.1) $$Q_t(x, \Gamma) = \text{Prob}\{X(t+\tau) \in \Gamma \mid X(\tau) = x\}$$

where $\Gamma \subset \Omega$ is a measurable subset of the state space

NB We assume independence of $\tau \Leftrightarrow$ system is autonomous

We also assume

(2.2) $$Q_t(x, \Omega) \leq 1$$

and, even though this isn't really necessary

(2.3) $$\lim_{t \downarrow 0} Q_t(x, \Gamma) = \delta_x(\Gamma) = \begin{cases} 1 & \text{if } x \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

and

(2.4) $$Q_{t+s}(x, \Gamma) = \int_\Omega Q_s(x, dz)\, Q_t(z, \Gamma), \quad s, t > 0$$

Chapman - Kolmogorov consistency required by the interpretation

1.3/ Take $\tau = 0$ in (2.1), but consider the situation that we do <u>not observe</u> where exactly the individual is and yet know

(2.5) $$\mu_0(\Gamma) = \text{Prob}\{X(0) \in \Gamma\}$$

Define likewise

(2.6) $$\mu_t(\Gamma) = \text{Prob}\{X(t) \in \Gamma\}$$

Then, according to (2.1)

(2.7) $$\mu_t(\Gamma) = \int_\Omega \mu_0(dz)\, Q_t(z,\Gamma)$$

which we write as (so (2.7) yields the definition of $T(t)$)

(2.8) $$\mu_t = T(t)\mu_0$$

<u>Exercise</u> Prove that $T(t+s) = T(t)T(s)$ $\quad t,s>0$ <u>semigroup</u>

<u>Interpretation</u>: μ_t provides a probabilistic description of the state of the individual, $T(t)$ describes the updating of this information, $T(0) = I$, $\{T(t)\}$ has the <u>semigroup</u> property, i.e., the <u>dynamical systems</u> property

maps may be <u>nonlinear</u>

Important special case: $Q_t(x,\cdot)$ has a <u>density</u>

(2.9) $$Q_t(x,\Gamma) = \int_\Gamma u(x,t,y)\,dy$$

$$u: \Omega \times \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$$

(2.10) $$u(x,t+s,y) = \int_\Omega u(x,s,z)\,u(z,t,y)\,dz \qquad \text{C-K}$$

(2.7) $$\mu_t(\Gamma) = \int_\Omega \mu_0(dz)\int_\Gamma u(z,t,y)\,dy = \int_\Gamma\left(\int_\Omega \mu_0(dz)\,u(z,t,y)\right)dy$$

1.4 / (2.11) $$(T(t)\phi)(x) = \int_\Omega \phi(z)\, u(z,t,x)\, dz$$

$\boxed{\text{independence}}$

<u>Re-interpretation</u> $\mu_t(\Gamma)$ is the fraction of a very large population of individuals having i-state belonging to Γ

<u>NB</u> If $Q_t(x,\Omega) < 1$ we interpret $1 - Q_t(x,\Omega)$ as the prob. of dying in the time interval $[0,t)$. The fraction is relative to the population alive at time $t=0$. So we could state:

$\mu_t(\Gamma)$ is the fraction of a very large population of individuals existing at time zero, that is alive at time t and has, at that time, i-state that belongs to Γ.

<u>NB</u> μ or ϕ now describes the <u>population state</u> p-state (see notes)

So far: 1. change of i-state, survival, but <u>no reproduction</u>

2. no interaction $\leftarrow$ <u>environmental condition</u> (wait till chemotaxis)

How the hell do you ever construct $Q_t(x,\Gamma)$ or $u(x,t,y)$? (having all these properties) The idea of a <u>generator</u>

<u>Discrete time</u>: $T(2) = T(1)\,T(1) = T(1)^{(2)}$

$T(n+1) = T(n)\,T(1) = T(1)\,T(n) = \dots = T(1)^{(n)}$

The map $T(1)$ <u>generates</u> the dynamics.

As a rule, <u>in continuous time</u> $T(t)\varphi \to \varphi$ as $t \downarrow 0$ (at least for a large class of initial data φ), so taking the limit only yields <u>system unspecific</u> information.

However, we might compute the <u>rate of change</u>

$$A\phi = \lim_{t\downarrow 0} \frac{1}{t}\left(T(t)\phi - \phi\right) \qquad D(A) = \{\phi : \text{limit exists}\}$$

1.5 So with $w(t) = T(t)\phi$ and $\phi \in D(A)$

we obtain the abstract ODE $\frac{dw}{dt} = Aw \quad , \quad w(0) = \phi$

which becomes a PDE if ϕ is a function of x and A involves x-derivatives.

Conversely, we may actually "derive" (i.e., motivate) A formally on the basis of modelling considerations and next solve $\frac{dw}{dt} = Aw$, $w(0) = \phi$ to define $T(t)$ ("autonomous" means now that A does not depend on t and the semigroup property then reflects the uniqueness of the solution). Chapter 3 provides an example

NB 1) contributions of different mechanisms to the rate of change can be added, in contrast to contributions to changes over a finite time interval

2) Many physical, chemical, biological ... laws do give expressions for the rate of change of quantities that describe the state

modelling considerations $\Rightarrow$ would-be generator

We postpone looking at $w(t,x) := (T(t)\phi)(x) = \int_\Omega \phi(z) u(z,t,x)\,dz$, cf. (2.11) and look instead at

$$(3.4) \qquad v(t,x) = \int_\Omega Q_t(x,dy)\,\psi(y) =: (T^*(t)\psi)(x)$$

NB $x \sim$ position at time 0 and $y \sim$ position at time t

Note that $v(t,x)$ is the expectation of $\psi(X(t))$, given $X(0) = x$

Here ψ is a given bounded continuous function on Ω.

By C-K (2.4)

$$(3.5) \qquad v(t+s,x) = \int_\Omega Q_s(x,dz)\,v(t,z)$$

1.6/ Now assume $Q_t(x,\Omega)=1$ and write

$$(3.6)\quad \frac{v(t+h,x)-v(t,x)}{h} = \frac{1}{h}\int_\Omega Q_h(x,dz)\left[v(t,z)-v(t,x)\right]$$

want to consider limit $h\downarrow 0$

So note: 1.) we compute the limit at t, not at time zero; since we look at an autonomous system, this makes no difference whatsoever. 2.) We make first a small time step h and then a big time step t, when decomposing $t+h$ into h and t

Assume that v is a smooth function of the spatial variable

Assume that the support of $Q_h(x,\cdot)$ may be large, but that most of the mass is concentrated near x, i.e., the likelihood of going very far is very small

Write

$$(*)\quad v(t,z)-v(t,x) = (z-x)\frac{\partial v}{\partial x}(t,x) + \frac{1}{2}(z-x)^2\frac{\partial^2 v}{\partial x^2}(t,x) + R(t,x,z)$$

Assume that $\forall \varepsilon>0$

i) $$\frac{1}{h}\int_{|z-x|\geq\varepsilon} Q_h(x,dz) \to 0 \quad \text{as } h\downarrow 0 \qquad (3.1)$$

ii) $$\frac{1}{h}\int_{|z-x|<\varepsilon} (z-x)\,Q_h(x,dz) \to b(x) \quad \text{as } h\downarrow 0 \qquad (3.2)$$

drift ~ mean

iii) $$\frac{1}{h}\int_{|z-x|<\varepsilon} (z-x)^2\,Q_h(x,dz) \to a(x) \quad \text{as } h\downarrow 0 \qquad (3.3)$$

variance

Inserting $(*)$ into (3.6) and taking the limit, assuming it exists, we find

$$\frac{\partial v}{\partial t}(t,x) = \frac{1}{2}a(x)\frac{\partial^2 v}{\partial x^2}(t,x) + b(x)\frac{\partial v}{\partial x}(t,x)$$

$$+ \lim_{h\downarrow 0}\frac{1}{h}\int_{\{z\,|\,|z-x|<\varepsilon\}} Q_h(x,dz)\,R(t,x,z)$$

write $\Omega = \{z\,|\,|z-x|<\varepsilon\} \cup \{z\,|\,|z-x|\geq\varepsilon\}$

1.7/ Assume $\forall \eta > 0 \ \exists \varepsilon = \varepsilon(x,\eta)$ such that

$$|R(t,x,z)| < \eta |z-x|^2 \quad \text{for } z \text{ with } |z-x|<\varepsilon$$

then

$$\left| \frac{1}{h} \int_{\{z \mid |z-x|<\varepsilon\}} Q_h(x,dz) R(t,x,z) \right| < \eta a(x) \quad \text{for small } h$$

So the limit of ↑ for $h \downarrow 0$ can be made as small as we wish, so has to be zero. We thus arrive at the <u>backward equation</u>

$$(3.7) \qquad \frac{\partial v}{\partial t} = \frac{1}{2} a \frac{\partial^2 v}{\partial x^2} + b \frac{\partial v}{\partial x}$$

By considering <u>all</u> bounded continuous functions ψ, the functions v fully characterize $Q_t(x,\cdot)$. Hence the solutions of (3.7) determine Q (recall our earlier remark: if we can choose a and b on the basis of modelling considerations, we can define the transition function if we can solve (3.7)).

Now assume (2.9), i.e., assume that Q has a density u, then (3.4) amounts to

$$(3.8) \qquad v(t,x) = \int_{\Omega} u(x,t,y)\psi(y)\,dy =: (T^*(t)\psi)(x)$$

For any L_1 function ϕ on Ω, define the pairing

$$(3.9) \qquad \langle \phi, \psi \rangle = \int_{\Omega} \phi(z)\psi(z)\,dz$$

Then

$$\langle \phi, v(t,\cdot) \rangle = \langle w(t,\cdot), \psi \rangle \quad \text{with}$$

$$(3.10) \qquad w(t,y) := \int_{\Omega} \phi(z) u(z,t,y)\,dz \underset{(2.11)}{\uparrow}{=} (T(t)\phi)(y)$$

In other symbols

$$(3.11) \qquad \langle \phi, T^*(t)\psi \rangle = \langle T(t)\phi, \psi \rangle$$

If in line with this we write $\quad (3.7) \quad \frac{dv}{dt} = A^* v$

we want to find A such that $\quad \frac{dw}{dt} = Aw$

This involves integration-by-parts and provided the boundary terms vanish, we find

$$(3.12) \qquad \frac{\partial w}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(a(y)w(t,y)\right) - \frac{\partial}{\partial y}\left(b(y)w(t,y)\right)$$

which is the Kolmogorov <u>forward equation</u> aka the <u>Fokker-Planck equation</u>.

Put $b=0$ and $a=2$, take $\Omega = [-1,+1]$ then the boundary terms are $\left[\phi\psi' - \phi'\psi\right]_{-1}^{+1}$

1.) $\phi(\pm 1) = 0 = \psi(\pm 1)$ absorbing boundary big monster

2.) $\phi'(\pm 1) = 0 = \psi'(\pm 1)$ reflecting boundary no-flux

next lecture

self-adjoint