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The zero set of a solution of a parabolic equation

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1. Introduction

In this note we study the zero set of a solution $u(t, x)$ of the equation

$$(1.1) \qquad u_t = a(x, t)\,u_{xx} + b(x, t)\,u_x + c(x, t)\,u.$$

Our motivation for doing so comes from a number of recent papers on the dynamics of semilinear analogues of (1. 1) (see [A], [AF], [BF1], [BF2], [BF3], [M1], [M2], [H2]). In one way or another all these papers use a result of the following kind:

Let u be a solution of (1. 1) on $Q = \{(x, t) : 0 \leqq x \leqq 1, 0 \leqq t \leqq T\}$ with Dirichlet boundary conditions: $u(0, t) \equiv u(1, t) \equiv 0$. Define the "number of zeroes" of $u(\cdot, t)$ to be the supremum over all k such that there exists $0 < x_1 < x_2 < \cdots < x_k < 1$ with

$$u(t, x_i) \cdot u(t, x_{i+1}) < 0 \qquad (i = 1, 2, \ldots, k-1).$$

Let $z(t)$ denote this supremum.

The result we mean is that $z(t)$ is a nonincreasing function of t. Loosely speaking, the number of zeroes of a solution of (1. 1) cannot increase with time.

Results of this nature were obtained by Nickel ([N]) in 1962, and revived by Matano ([M2]) in 1982 and Henry ([H2]) in 1985.

None of the existing results deals with the actual number of zeroes of $u(\cdot, t)$. Instead, an alternative definition of this number is usually given (like the definition of $z(t)$ we just gave). Moreover the possiblility that $z(t) = +\infty$ for $0 \leqq t \leqq T$ is never excluded. There is therefore some incompleteness in the existing results on $z(t)$.

The effect of this incompleteness is that in some of the applications to semilinear equations extra hypotheses have to be included so as to ensure $z(t) < \infty$ (e.g. piecewise monotonicity of solutions, or even real analyticity of all occurring functions as in [AF]).

Here we present a detailed description of the zero set of a solution of (1. 1) under very general assumptions on the coefficients a, b and c.

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In addition to the application to the dynamics of semilinear parabolic equations, the results of this paper may also be applied to other problems in PDE. One of these is described at the end of this introduction, and may be referred to as "time dependent Sturm Liouville theory". Another one is the "geometric heat equation", or the "curve shrinking problem", which was considered by Grayson ([Gr]). The nature of these applications is described in [A'].

The main part of this paper deals with solutions of

$$u_t = u_{xx} + q(x, t)u \qquad x \in \mathbb{R}, \quad 0 < t < T \tag{1. 2}$$

where $q \in L_\infty$ and $|u(x, t)| \leqq A e^{Bx^2}$ for some constants A and B.

Our main results concerning $u(x, t)$ are the following.

Theorem A. *For each $t \in (0, T)$ the zero set of $u(\cdot, t)$,*

$$Z_t = \{x \in \mathbb{R} | u(x, t) = 0\},$$

is a discrete subset of $\mathbb{R}$.

Theorem B. *If at (x_0, t_0) both u and u_x vanish then there is a neighbourhood $N = [x_0 - \varepsilon, x_0 + \varepsilon] \times [t_0 - \delta, t_0 + \delta]$ of (x_0, t_0) such that*

(i) $u \neq 0$ *on the sides of N* (i.e. $u(x_0 \pm \varepsilon, t) \neq 0$ *for* $|t - t_0| \leqq \delta$),

(ii) $u(\cdot, t + \delta)$ *has at most one zero in the interval* $[x_0 - \varepsilon, x_0 + \varepsilon]$,

(iii) $u(\cdot, t - \delta)$ *has at least two zeroes in the interval* $[x_0 - \varepsilon, x_0 + \varepsilon]$.

This theorem says, roughly, that if, at $t = t_0$, $u(\cdot, t)$ has a multiple zero then, for $t < t_0$, $u(\cdot, t_0)$ has more zeroes than for $t > t_0$.

In case the coefficients and the solution are real analytic theorem A is trivial and theorem B was already proved in [AF], theorem 5. 1.

These results are also true for solutions of (1. 1) instead of (1. 2) (with the same a priori bound $|u(x, t)| \leqq A \exp(Bx^2)$), if one assumes the following about the coefficients a, b and c:

$$a,\ a^{-1},\ a_t,\ a_x \ \text{ and } \ a_{xx} \in L_\infty, \tag{1. 3.a}$$

$$b,\ b_t \ \text{ and } \ b_x \in L_\infty, \tag{1. 3.b}$$

$$c \in L_\infty. \tag{1. 3.c}$$

Furthermore a should be positive, so that the equation is parabolic. The reason that theorems A and B hold for solutions u of (1. 1) is that (1. 1) can be reduced to an equation of the type (1. 2). This reduction proceeds in two steps. First introduce a new coordinate

$$y = \int_0^x a(s, t)^{-\frac{1}{2}} ds$$

In the y, t coordinates u satisfies

$$u_t = u_{yy} + \tilde{b}(y, t) u_y + \tilde{c}(y, t) u$$

where $\tilde{b}$ and $\tilde{c}$ satisfy (1. 3b, c). Next substitute

$$v(y, t) = \exp\left[\frac{1}{2}\int_0^y \tilde{b}(s, t)\, ds\right] u(y, t).$$

Then v satisfies $v_t = v_{yy} + q(y, t) v$ for suitable q. It is not hard to verify that q is bounded, and that if u satisfies an a priori bound of the form $|u| \leqq A \exp(Bx^2)$ then so does v (with different constants A and B). Clearly u and v have the same zero set, and theorems A and B are invariant under the coordinate change $(x, t) \to (y, t)$.

The theorems A and B also hold for solutions of (1. 2) on bounded domains, if one imposed either Dirichlet, Neumann or periodic boundary conditions.

To see why this is so, we first consider a bounded solution $u(x, t)$ of (1. 2) on the rectangle $[0, 1] \times [0, T)$, which satisfies Dirichlet boundary conditions, i.e.

$$u(0, t) \equiv u(1, t) \equiv 0 \qquad (0 \leqq t \leqq T).$$

Let $U(x, t)$ be the (unique) extension of $u(x, t)$ to $\mathbb{R} \times [0, T]$ which satisfies

$$U(-x, t) \equiv -U(x, t) \equiv U(2-x, t).$$

(Note that this implies $U(x+2, t) \equiv U(x, t)$.) Then U satisfies

$$U_t = U_{xx} + Q(x, t) U$$

where $Q(x, t)$ is the extension of $q(x, t)$ which satisfies $Q(x+1, t) \equiv Q(x, t)$.

Since both U and Q are bounded we can apply theorems A and B to U.

If u satisfies Neumann boundary conditions then one should require U to be even instead of odd.

The case of periodic boundary conditions is a special case of the situation on $\mathbb{R} \times (0, T)$.

Next we show how equation (1. 1) can be reduced to (1. 2) if the domains are bounded.

Let $u(x, t)$ be a bounded solution of (1. 2) on $[0, 1] \times [0, T]$ and assume that the coefficients satisfy (1. 3.a, b, c). Then define

$$\alpha(t) = \int_0^1 a(x, t)^{-\frac{1}{2}} dx,$$

$$y(x, t) = \frac{1}{\alpha(t)} \int_0^x a(\xi, t)^{-\frac{1}{2}} d\xi,$$

$$s(t) = \int_0^t \alpha(t)^2 dt.$$

The transformation $(x, t) \to (y, s)$ maps $[0, 1] \times [0, T]$ to $[0, 1] \times [0, S]$ for some $S > 0$, and is $C^{1,1}$ in the x-coordinate and $C^{0,1}$ in the t-coordinate.

In the new coordinates u satisfies

$$u_s = u_{yy} + \tilde{b}(s, y) u_y + \tilde{c}(s, y) u.$$

As before, we can get rid of the first term by multiplying u with a suitable exponential. In this way we get a solution v of $v_s = v_{yy} + q \cdot v$ with Dirichlet boundary data. Again we can apply theorems A and B.

We summarise this discussion in the following theorem.

Theorem C. *Let* $u : [0, 1] \times [0, T] \to \mathbb{R}$ *be a bounded solution of* (1. 1) *which satisfies either Dirichlet, Neumann or periodic boundary conditions.*

Assume that a, b and c satisfy (1. 3.a, b, c), *and in addition, in the case of Neumann boundary conditions, assume that* $a \equiv 1$ *and* $b \equiv 0$.

Let $z(t)$ *denote the number of zeroes of* $u(\cdot, t)$ *in* $[0, 1]$. *Then*

(a) *for* $t > 0$ $z(t)$ *is finite,*

(b) *if* (x_0, t_0) *is a multiple zero of u then for all* $t_1 < t_0 < t_2$ *we have* $z(t_1) > z(t_2)$.

Finally we observe that if inhomogeneous boundary conditions are imposed, we can still say something about the zero set of $u(x, t)$.

Theorem D. *Let* $u: [x_0, x_1] \times [0, T] \to \mathbb{R}$ *be a classical continuous solution of* (1. 1) *such that*

$$u(x_i, t) \neq 0 \qquad (i = 1, 2; 0 \leqq t \leqq T).$$

Assume that a, b and c satisfy (1. 3.a, b, c). *Then statements* (a) *and* (b) *of theorem* C *hold in this case.*

Here "classical" means that u_t, u_x and u_{xx} are continuous on $(x_0, x_1) \times [0, T]$.

To prove this theorem observe that we can extend our solution u to a bounded function on the strip $\mathbb{R} \times [0, T]$, whose absolute value is bounded from below outside the rectangle $[x_0, x_1] \times [0, T]$. Moreover we can modify this function outside a neighborhood of its zero set so that u_t, u_x, u_{xx} become bounded functions. Next we extend the coefficients $a(x, t)$ and $b(x, t)$ to $\mathbb{R} \times [0, T]$ in such a way that they still satisfy (1. 3.a, b) and define

$$\tilde{c}(x, t) = (u_t - a u_{xx} - b u_x)/u.$$

Since $\tilde{c}$ satisfies (1. 3c) we can apply our previous results to the new function u, to prove theorem D.

As an application of these results we mention the following.

Let $p(x, t)$, $q(x, t)$ be two doubly periodic functions (i.e.

$$q(x+1, t) \equiv q(x, t+1) \equiv q(x, t)$$

and the same for p) of which q is bounded and p is C^1.

Consider the operator L defined by

$$L=\frac{\partial}{\partial t}-\left(\frac{\partial}{\partial x}\right)^2-p(x,t)\frac{\partial}{\partial x}-q(x,t),$$

and let $\Sigma\subset\mathbb{C}$ be the set of numbers λ for which there exists a bounded, complex valued, solution $u(x, t)$ of

$$\begin{aligned}&Lu=0,\\&u(x+1,t)=u(x,t)\qquad x,t\in\mathbb{R},\\&u(x,t+1)=\lambda\cdot u(x,t)\qquad x,t\in\mathbb{R},\end{aligned}$$

It follows from parabolic regularity theory that Σ is a countable set of points with $\lambda=0$ as only limit point. So we can write Σ as $\{\lambda_n\}_{n\geqq 0}$ where λ_n occurs as often as its multiplicity, and $|\lambda_n|\geqq|\lambda_{n+1}|$.

In this setting the following holds:

$$\cdots|\lambda_{2n}|>|\lambda_{2n+1}|\geqq|\lambda_{2n+2}|>|\lambda_{2n+3}|\cdots.$$

In particular, the multiplicity of an eigenvalue never exceeds 2. The proof of this fact was given in section 2 of [AF], using the assumption that p and q are real analytic. Given theorem C of this note one can prove the general case in exactly the same way.

The proof of our main result (theorems A and B) is based upon a coordinate transformation. The transformation in question is

$$(u,x,t)\longrightarrow(w,\xi,\tau)$$

where

$$t=t_0-e^{-2\tau},\ x=x_0+e^{-\tau}\xi$$

and

$$w=\exp\left(-\frac{1}{2}\xi^2\right)u.$$

It turns out that w satisfies a nice parabolic equation in the ξ, τ coordinates. By studying the asymptotic behaviour of w as $\tau\longrightarrow\infty$ (i.e. of u as $t\uparrow t_0$) we get some information about the zero set of u near (x_0, t_0). This is done in sections 2, 3 and 4. Then some additional arguments are needed to prove theorems A and B. These arguments are presented in section 5.

Before I start with the main portion of this paper, I would like to thank Lillian Chappelle for typing the manuscript.

2. Notation

Let $u(x, t)$ be a nonzero solution of the parabolic equation

(2. 1) $$u_t-u_{xx}=q(x,t)u\qquad x\in\mathbb{R},\quad 0<t<T.$$

We assume that u satisfies the following estimate

$$|u(x,t)| \leqq K e^{Lx^2}$$

for all $(x,t) \in Q = \mathbb{R} \times (0,T)$ and that the coefficient $q(x,t)$ belongs to $L_\infty(Q)$. By rescaling (i.e. taking $(\varepsilon x, \varepsilon^2 t)$ as new coordinates) and passing to a multiple of u we may assume that

$$(2.2) \qquad \begin{aligned} |u(x,t)| &\leqq e^{x^2/16} \quad \text{(a.e.)}, \\ 2|q(x,t)| &\leqq 1 \quad \text{(a.e.)}. \end{aligned}$$

It follows from local regularity theory of the inhomogeneous heat equation that u and u_x are locally Hölder continuous functions in Q, of any exponent $\alpha < 1$.

We define Z to be the *zero set* of u, i.e.

$$Z = \{(x,t) \in Q : u(x,t) = 0\},$$

and S to be the *singular part* of Z,

$$S = \{(x,t) \in Z : u_x = 0\}.$$

The complement of S in Z, denoted by R, will be called the *regular part* of Z.

It follows from the implicit function theorem that R is the disjoint union of curves $C_i \subset Q$. Each C_i is given by the graph of a Hölder continuous function:

$$C_i : x = \gamma_i(t) \qquad (t_{i,-} < t < t_{i,+}).$$

In order to study the local structure of S we introduce the functions $w(x,t,h,\xi)$. They are defined by

$$w(x,t,h,\xi) = e^{-\frac{1}{2}\xi^2} u(x + 2h\xi, t).$$

It follows from our estimate (2.2) that for all $(x,t) \in Q$ and $0 < h \leqq 1$ the function $w(x,t,h,\cdot)$ belongs to $L_2(\mathbb{R})$. Moreover, as elements of $L_2(\mathbb{R})$ they depend continuously on x, t and h.

Whenever $w(x,t,h,\cdot)$ is not identically zero we define

$$W(x,t,h,\xi) = \frac{w(x,t,h,\xi)}{\|w(x,t,h,\cdot)\|_{L_2}}.$$

3. Basic estimates

For given $(x,t) \in Q$ we consider

$$\begin{aligned} v(\tau,\xi) &= w(x, t - e^{-2\tau}, e^{-\tau}, \xi) \\ &= e^{-\frac{1}{2}\xi^2} u(x + 2e^{-\tau}\xi, t - e^{-2\tau}). \end{aligned}$$

This function is defined for $\tau \geqq -\frac{1}{2}\log(t) = \tau_0$. A straightforward computation shows that v satisfies

$$v_\tau = \frac{1}{2} v_{\xi\xi} - \frac{1}{2}(\xi^2 - 1)v + 2e^{-2\tau} q v, \tag{3. 1}$$

where q should be read as $q(x + 2e^{-\tau}\xi, t - e^{-2\tau})$.

Since $v(\tau, \cdot)$ belongs to $L_2(\mathbb{R})$ we shall reformulate (3. 1) in an $L_2(\mathbb{R})$-setting. Define the following Hilbert spaces

$$E_0 = L_2(\mathbb{R}),$$
$$E_1 = \{u \in L_2(\mathbb{R}) \mid x^2 u,\ u_{xx} \in L_2(\mathbb{R})\},$$
$$E_{\frac{1}{2}} = \{u \in L_2(\mathbb{R}) \mid xu,\ u_x \in L_2(\mathbb{R})\},$$

and let the operators A and $B(\tau)$ be given by

$$Av = -\frac{1}{2} v_{\xi\xi} + \frac{1}{2}(\xi^2 - 1)v,$$
$$B(\tau) \cdot v = 2e^{-2\tau} q \cdot v.$$

Then $A: E_1 \to E_0$ is bounded. Seen as an unbounded operator in E_0 it is self adjoint with respect to the usual innerproduct on $L_2(\mathbb{R})$. Its spectrum consists of the simple eigenvalues $\{0, 1, 2, \ldots\}$. The eigenfunctions have the form

$$\varphi_n(\xi) = e^{-\frac{1}{2}\xi^2} H_n(\xi)$$

where H_n is a multiple of the n-th Hermite polynomial. These facts are well known since A is the quantum mechanical Hamiltonian for the harmonic oscillator (see [RS], appendix to V.3 on page 141).

The operators $B(\tau)$ are bounded on E_0. In view of (2. 2) we even have

$$\|B(\tau)\|_{E_0, E_0} \leqq e^{-2\tau}. \tag{3. 2}$$

With these definitions (3. 1) can now be written as

$$v_\tau + Av = B(\tau)v \qquad \tau > \tau_0, \tag{3. 3}$$

where v is a mild solution of (3. 3). This means that for $\tau_0 < \tau_1 < \tau_2$ one always has

$$v(\tau_2) = e^{(\tau_1 - \tau_2)A} v(\tau_1) + \int_{\tau_1}^{\tau_2} e^{(t - \tau_2)A} B(\tau)\, v(\tau)\, dt.$$

(the "variation of constants formula").

Using the bound $\|B(\tau)\| \leqq e^{-2\tau}$, and the fact that A generates a contraction semigroup, we find a bound for $\|v(\tau)\|$. Indeed, by applying a Picard iteration to the variation of constants formula, and observing that

$$\int_0^\infty \|B(\tau)\| \, d\tau = 1/2 < 1$$

one finds

$$\|v(\tau_2)\| \leqq 2\,\|v(\tau_1)\| \qquad (\forall \tau_2 \geqq \tau_1).$$

One can amplify this estimate by observing that if $0 \leqq \theta < 1$,

$$\|A^\theta e^{tA}\| \leqq C \cdot t^{-\theta} \qquad (0 \leqq t < \infty)$$

(where C depends on θ only). This, and the variation of constants formula lead to

$$\|A^\theta v(\tau_2)\| \leqq C\{1 + (\tau_2 - \tau_1)^{-\theta}\}\,\|v(\tau_1)\|. \tag{3. 4}$$

If we apply this estimate to the functions w, then we get the following lemma.

Lemma 3. 1. *For all (x, t) in Q and $h \leqq 1$ for which $h^2 < t < T - h^2$ there is*

$$w(x, t, h, \cdot) \in E_\theta \qquad (0 \leqq \theta < 1)$$

and $w(x, t, h, \cdot)$ depends continuously on (x, t, h) in the E_θ topology.

Here E_θ denotes $\mathrm{dom}(A^\theta)$ $(0 \leqq \theta < 1)$.

Proof. Consider

$$v(\tau, \xi) = w(x, t + h^2 - e^{-2\tau}, e^{-\tau}, \xi),$$

and define

$$\tau_1 = -\frac{1}{2}\log(2h^2), \; \tau_2 = -\log(h).$$

Then

$$v(\tau_2, \cdot) = w(x, t, h, \cdot)$$

and $\tau_2 - \tau_1 = \frac{1}{2}(\log 2h^2)$ is a constant. Therefore the inequality (3. 4) implies $w \in E_\theta$.

To prove continuous dependence, let (x_n, t_n, h_n) converge to (x, t, h), and let w_n and w be the corresponding functions. Our estimates imply that the w_n remain bounded in all E_θ spaces. Since A has compact resolvent this means that the sequence w_n is precompact in any E_θ. We already know that $w_n \to w$ in E_0, so that w is the only possible limit point in the E_θ-topologies. Hence $w_n \to w$ in E_θ.

4. Compactness

In this section we shall prove the following two results.

Lemma 4. 1. *For all x, t and h, one has*

$$\|w(x, t, h, \cdot)\|_{E_\theta} \neq 0.$$

As a consequence of this lemma the function $W(x, t, h, \cdot)$ is always well defined. Let $K \subset Q$ denote the compact set

$$K = [-x_0, x_0] \times [2h_0^2, T - 2h_0^2]$$

for some $x_0 > 0$ and $h_0 > 0$.

Theorem 4.2. *The set*

$$\{W(x, t, h, \cdot) \mid 0 < h \leqq h_0 \quad \text{and} \quad (x, t) \in K\}$$

is precompact in E_θ for any $\theta < 1$.

Proof of Lemma 4.1. Whether or not $w(x, t, h, \cdot) \equiv 0$ does not depend on x or h. Let t_0 be the smallest $t > 0$ such that $w(x, t, h, \cdot)$ vanishes, and assume that $t < T$. Let $h = 1/3$ and consider

$$v(\tau, \xi) = w(x, t_0 + h^2 - e^{-2\tau}, e^{-\tau}, \xi).$$

Then v is a mild solution of (3.3). Since the gaps in the spectrum of A all have length one, and $\|B(\tau)\| \leqq e^{-2\tau} \leqq 2h^2 < \frac{1}{2}$ for $\tau \geqq -\log(h\sqrt{2})$, the results in [A] imply that $v(\tau, \cdot)$ is nonzero for all τ, unless $v(-\log(h\sqrt{2}), \cdot)$ vanishes identically (following a suggestion of the referee we have included the relevant statements of [A] in the appendix to this paper).

Therefore $v(\log h, \cdot) = w(x, t_0, h, \cdot) \neq 0$ which contradicts our definition of t_0.

Now we turn to the proof of the theorem. Our proof is based on the following observation. Let C be a closed subset of the unit ball in E_0. Denote the orthogonal projection onto the first n eigenfunctions of A by P_n, and write Q_n for $1_{E_0} - P_n$. Suppose the set C is such that, for any $\varepsilon > 0$ there exists an n_ε so that

$$\|Q_{n_\varepsilon} w\| \leqq \varepsilon \|P_{n_\varepsilon} w\| \tag{4.1}$$

holds for all w in C. Then C is compact.

Indeed, the condition (4.1) implies that C is contained in an ε-neighbourhood of the unit ball in the range of P_{n_ε}. Since P_{n_ε} has finite rank this unit ball is compact, so that C lies in an ε-neighbourhood of a compact set, for any $\varepsilon > 0$. Thus for given $\varepsilon > 0$ we can find a finite number of balls $B(x_1, \varepsilon), \ldots, B(x_m, \varepsilon)$ whose union contains the unit ball in the range of P_{n_ε}, and whose centres lie in this unit ball. The doubled balls $B(x_j, 2\varepsilon)$ will then cover our set C, by construction. We can do this for any $\varepsilon > 0$ so that C is totally bounded. Since C is complete as a metric space, it is also compact.

With this compactness criterion in mind we define $N_\varepsilon(x, t, h)$ to be the smallest integer n such that

$$\|Q_n W(x, t, h, \cdot)\| \leqq \varepsilon \|P_n W(x, t, h, \cdot)\|$$

holds, and we try to estimate $N_\varepsilon(x, t, h)$. Since the projections P_n converge strongly to 1_{E_0} and the Q_n tend to zero, the number $N(x, t, h)$ is always finite.

A better estimate is the following.

Lemma 4. 3. *If* $\varepsilon < 2h_0 \leqq 1$ *then*

$$\sup(N_\varepsilon(x, t, h) : (x, t) \in K, \ 0 < h \leqq h_0) \tag{4. 2}$$
$$\leqq \sup\left(N_\varepsilon(x, t, h) : (x, t) \in K_1 \ \text{ and } \ \frac{1}{2}\varepsilon \leqq h \leqq h_0\right)$$

where

$$K_1 = \{(x, t) \mid -x_0 \leqq x \leqq x_0, \ h_0^2 \leqq t \leqq T - h_0^2\}.$$

This lemma directly implies theorem 4. 2. To see this observe that the second supremum in lemma 4. 3 is taken over a compact set of parameters (in (x, t, h)-space). Since the $W(x, t, h, \cdot)$ depend continuously on (x, t, h) this supremum therefore is finite. Lemma 4. 3 tells us that the set of $W(x, t, h, \cdot)$ with $(x, t) \in K$ and $h \in (0, h_0)$ satisfies (4. 1), so that it is precompact in E_0. The smoothing property of the parabolic equation then implies that this set is also precompact in E_θ for $\theta < 1$. So it remains to prove the lemma.

Proof of Lemma 4. 3. Define

$$v(\tau, \xi) = w(x, t - e^{-2\tau}, e^{-\tau}, \xi),$$

$$F(\tau) = \frac{1}{2}(\varepsilon^2 \|P_n v(\tau)\|^2 - \|Q_n v(\tau)\|^2).$$

Then $F(\tau) \geqq 0$ is equivalent to

$$\|Q_n W(x, t - e^{-2\tau}, e^{-\tau}, \cdot)\| \leqq \varepsilon \|P_n W(\cdots)\|.$$

Since the $v(\tau)$ are absolutely continuous in τ we have:

$$\begin{aligned} F'(\tau) &= \varepsilon^2(-Av + Bv, P_n v) - (-Av + Bv, Q_n v) \\ &= -\varepsilon^2(Av, P_n v) + (Av, Q_n v) - (Bv, (Q_n - \varepsilon^2 P_n) v) \\ &\geqq -n\varepsilon^2 \|P_n v\|^2 + (n+1)\|Q_n v\|^2 - e^{-2\tau}\|v\|^2 \\ &= -(n+1)F(\tau) + \varepsilon^2 \|P_n v\|^2 - e^{-2\tau}\|v\|^2. \end{aligned}$$

Therefore

$$\frac{d}{d\tau}(e^{(n+1)\tau}F(\tau)) \geqq \varepsilon^2 \|P_n v\|^2 - e^{-2\tau}\|v\|^2.$$

Now suppose that, at $\tau_0 = -\log\left(\frac{\varepsilon}{2}\right)$, $F(\tau_0)$ is positive. Then, by continuity of F, there is a largest interval $[\tau_0, \tau_1)$ on which F is positive. Positivity of F implies

$$\varepsilon^2 \|P_n v\|^2 \geqq \|Q_n v\|^2 = \|v\|^2 - \|P_n v\|^2$$

so that for $\tau \in [\tau_0, \tau_1]$ we have

$$\|P_n v\|^2 \geqq (1+\varepsilon^2)^{-1} \|v\|^2 \geqq \frac{1}{2} \|v\|^2,$$

and

$$\frac{d}{d\tau}(e^{(n+1)\tau} F(\tau)) \geqq \left(\frac{1}{2}\varepsilon^2 - e^{-2\tau}\right) \|v\|^2 \geqq 0.$$

So we see that if $\varepsilon > 2e^{-\tau_0}$ then $e^{(n+1)\tau} F(\tau)$ is nondecreasing after τ_0.

To complete the proof of Lemma 4. 3 we choose a point $(x_0, \tau_0) \in K$, an $h > 0$ and an $\varepsilon > 0$. If $h \geqq \frac{1}{2}\varepsilon$ then $N_\varepsilon(x_0, t_0, h)$ does not exceed the righthand side in (4. 2).

Next consider the case $h < \frac{1}{2}\varepsilon$. The function $F(\tau)$ which we have just introduced depends on the chosen point (x, t) (namely, through the definition of $v(\tau, \xi)$). If we choose this point to be $(x_0, t_0 + h^2)$, then the preceding discussions about $F(\tau)$ imply that

$$N_\varepsilon(x_0, t_0, h) \leqq N_\varepsilon\left(x_0, t_0 + h^2 - \left(\frac{\varepsilon}{2}\right)^2, \frac{\varepsilon}{2}\right).$$

So in this case $N_\varepsilon(x_0, t_0, h)$ is also dominated by the righthand side of (4. 2).

5. Consequences of compactness

The easiest consequence of theorem 4. 2 is the following

Lemma 5. 1. *The solution $u(x, t)$ of* (2. 1) *does not vanish on any interval* $(x_0, x_1) \times \{t_0\}$.

Proof. Suppose $u(x, t_0) \equiv 0$ for $x_0 < x < x_1$. Choose an $x_2 \in (x_0, x_1)$ and consider the functions $W(x_2, t_0, h)$ for small h. As $h \downarrow 0$ these functions converge pointwise to zero. But by theorem 4. 2 we can extract a subsequence which converges, in $L_2(\mathbb{R})$, to a function with norm one.

This is a contradiction, so that $u(x, t_0)$ cannot be zero on (x_0, x_1) after all.

This lemma gives us some information about how much curves in the zero set of u can "zig-zag".

To be more precise, let $\gamma(t)$ be a continuous function of t for $t_0 < t < t_1$, such that $u(\gamma(t), t) \equiv 0$. As an example one can take one of the curves in the regular part of the zero set. However, we do not exclude curves passing through the singular part S of Z, as long as they are graphs of continuous functions of t.

Lemma 5. 2. *If $t_0 > 0$ then $\lim_{t \downarrow t_0} \gamma(t)$ exists.*

Proof. Let J be the set of limit points of $\gamma(t)$ as $t \downarrow t_0$. If J is nonempty then the continuity of γ implies that J is connected. Since $u(x, t_0) = 0$ for all x in J, J cannot be an interval, by lemma 5. 1. Therefore J consists of one point only: the limit of $\gamma(t)$ as $t \downarrow t_0$.

To complete the proof we have to exclude the case that J is empty. If J is empty, then $\gamma(t) \to +\infty$ or $-\infty$ as $t \downarrow t_0$. Suppose that $\gamma(t) \to +\infty$. Then consider the region $G = \{(x, t) : t_0 < t < t_1, x > \gamma(t)\}$, lying to the right of the curve $\gamma(t)$. On the parabolic boundary of G our function u vanishes.

The maximum principle then implies that u vanishes in G. Since we are dealing with weak solutions on an unbounded domain we must be more precise here. To prove that u actually vanishes in G, consider

$$v(\xi, \tau) = e^{-\frac{1}{2}\xi^2} u(x_0 + 2e^{-\tau}\xi, t_0 + h^2 - e^{-2\tau})$$

where $h \leqq 1$, and $\tau \geqq -\log(h)$ (the value of x_0 may be chosen arbitrarily). Then v satisfies equation (3. 1).

Next define

$$M(\tau) = \int_{-\infty}^{\delta(\tau)} |v(\xi, \tau)|^2 d\xi$$

where $x_0 + 2e^{-\tau} \cdot \delta(\tau) = \gamma(t_0 + h^2 - e^{-2\tau})$ defines the quantity $\delta(\tau)$. Using equation (3. 1) one proves, by integration by parts, that

$$M'(\tau) \leqq C \cdot M(\tau)$$

for some constant C, independent of the solution u. Now, by assumption,

$$\lim_{t \downarrow t_0} \gamma(t) = -\infty \quad \text{i.e.} \quad \lim_{\tau \downarrow -\log h} \delta(\tau) = -\infty$$

so that $M(\tau) \to 0$ as $\tau \downarrow -\log(h)$, and thus $M(\tau)$ vanishes for all $\tau \geqq 0$, which, in turn, implies that $u(t, x)$ vanishes in G, for $t_0 < t \leqq t_0 + h^2$. Iteration of this argument then shows that u vanishes in all of G.

The upshot is that u vanishes on an interval in G, thereby contradicting lemma 5. 1. In the same way the assumption $\gamma(t) \to +\infty$ as $t \downarrow t_0$ also leads to a contradiction. This completes the proof of lemma 5. 2.

Next suppose we have two curves $\gamma_1(t)$, $\gamma_2(t)$ in the zero set of u, defined for $t_0 \leqq t \leqq t_1$, and suppose $\gamma_1(t_1) < \gamma_2(t_1)$. If for some $t < t_1$, $\gamma_1(t) = \gamma_2(t)$ occurs, then there is a largest t for which this happens. Call this moment t_2, and consider the region

$$G = \{(x, t) : t_2 < t < t_1, \gamma_1(t) < x < \gamma_2(t)\}.$$

Then u vanishes on the parabolic boundary of G, and the maximum principle implies that u vanishes on all of G. Since this contradicts lemma 5. 1 we have now shown

Lemma 5. 3. *Let $x = \gamma_k(t)$ $(k = 1, 2, t_0 \leqq t \leqq t_1)$ be two continuous curves in the zero set of u, such that $\gamma_1(t_1) < \gamma_2(t_1)$. Then $\gamma_1(t) < \gamma_2(t)$ for all $t \in [t_0, t_1]$.*

In the next lemma we show how curves in the zero set Z can be constructed.

Lemma 5. 4. *Let $(x_0, t_0) \in Z$ be given. Then there exists at least one continuous curve $\gamma(t)$ in Z defined on some nonempty interval $[t_1, t_0]$ such that $\gamma(t_0) = x_0$.*

Proof. Let $(x_0, t_0) \in Z$ be given. Consider $v(\tau, \xi)$ defined by

$$v(\tau, \xi) = w(x_0, t_0 - e^{-2\tau}, e^{-\tau}, \xi).$$

Then v satisfies (3. 1), or equivalently (3. 3). The discussion in [A], appendix, or [H2] on the asymptotic behaviour of solutions of (3. 3) shows that, as $\tau \to \infty$,

$$\frac{v(\tau, \cdot)}{\|v(\tau, \cdot)\|_{L_2}} \to \pm \varphi_n \tag{5. 1}$$

for some $n \geqq 0$. Hence φ_n denotes the n-th eigenfunction of the operator A of section 3.

For the moment suppose that $n \geqq 1$. Then φ_n has $n \geqq 1$ simple zeroes. Since the convergence in (5. 1) takes place in E_θ for all $\theta < 1$, and therefore in $C^1(\mathbb{R})$, the function v has at least n curves $\xi = \xi_k(\tau)$ $(\tau_1 \leqq \tau < \infty)$ in its zero set (where $k = 1, \ldots, n$ and τ_1 should be chosen sufficiently large). Therefore the curves

$$x = \gamma_k(t) = x_0 + \sqrt{t_0 - t}\; \xi_k(-\log \sqrt{t_0 - t})$$

with $t_0 - e^{-2\tau_1} \leqq t < t_0$ lie in the zero set of $u(x, t)$. Since $\gamma_k(t) \to x_0$ as $t \uparrow t_0$ the lemma is true in this case.

It should be noted that $v(\xi, \tau)$ could have more curves $\xi(t)$ in its zero set. For these other curves $\xi(\tau)$ would be unbounded as $\tau \uparrow \infty$, and our asymptotic description of $v(\xi, \tau)$ is not good enough to say anything about them.

We are left with the possibility that $n = 0$. Since the convergence in (5. 1) holds pointwise, and $v(\tau, 0) = u(x_0, t_0 - e^{-2\tau})$ tends to zero as $\tau \to \infty$, we must have $\|v(\tau, \cdot)\| \to 0$ (remember that $\varphi_0(\xi)$ is strictly positive).

The arguments in [A], [H2] show that the norm of $v(\tau, \cdot)$ cannot decay exponentially (this would imply $n \geqq 1$). In the following we show that v cannot decay at all, if $n = 0$, thus reaching a contradiction which excludes the case $n = 0$.

Let $S(\tau, \tau_0)$, $(\tau_0 < \tau)$ denote the evolution operator for the equation (3. 3). It is determined by the equation

$$S(\tau, \tau_0) = e^{(\tau_0 - \tau)A} + \int_{\tau_0}^{\tau} e^{(\sigma - \tau)A} B(\sigma)\, S(\sigma, \tau_0)\, d\sigma.$$

Define $R(\tau, \tau_0)$ to be $S(\tau, \tau_0) - e^{(\tau_0 - \tau)A}$. Then $R(\tau, \tau_0)$ satisfies

$$R(\tau, \tau_0) = \int_{\tau_0}^{\tau} e^{(\sigma - \tau)A} B(\sigma)\, e^{(\tau_0 - \sigma)A}\, d\sigma + \int_{\tau_0}^{\tau} e^{(\sigma - \tau)A} B(\sigma)\, R(\sigma, \tau_0)\, d\sigma.$$

Using our exponential estimate for B, we see that

$$(5.2) \qquad \|R(\tau, \tau_0)\| \leqq \frac{1}{2} e^{-2\tau_0} + \int_{\tau_0}^{\tau} e^{-2\sigma} \|R(\sigma, \tau_0)\| \, d\sigma.$$

We claim that this implies $\|R(\tau, \tau_0)\| \leqq e^{-2\tau_0}$. Indeed, for $\tau = \tau_0$ this is certainly true. Let τ_1 be the supremum of all $\tau > \tau_0$ for which the assertion is true. By continuity we have $\tau_1 > \tau_0$. If $\tau_1 < \infty$,then our inequality (5. 2) implies

$$\begin{aligned} \|R(\tau_1, \tau_0)\| &\leqq \frac{1}{2} e^{-2\tau_0} + \frac{1}{2} e^{-2\tau_0} \sup_{\tau < \tau_1} \|R(\tau, \tau_0)\| \\ &\leqq \frac{1}{2} (1 + e^{-2\tau_0}) e^{-2\tau_0} \\ &< e^{-2\tau_0} \end{aligned}$$

since $\tau_0 > 0$. By continuity this implies that the assertion is true for τ slightly larger than τ_1. The contradiction shows that τ_1 cannot be finite.

Let P be the orthogonal projection onto the lowest eigenvector of A, and write Q for $1 - P$. Then $\|P - e^{-\sigma A}\| = e^{-\sigma}$, so that

$$\begin{aligned} \|S(\tau, \tau_0) - P\| &= \|e^{(\tau_0 - \tau)A} - P + R(\tau, \tau_0)\| \\ &\leqq e^{-(\tau - \tau_0)} + e^{-2\tau_0}. \end{aligned}$$

Now let τ_0 and τ be so large that

$$\|S(\tau, \tau_0) - P\| \leqq 10^{-2},$$

and $\|Q v(\tau_0)\| \leqq 10^{-2} \|P v(\tau_0)\|$ hold. This is possible since $v/\|v\|$ converges to φ_0 as $\tau \longrightarrow \infty$.

Then, using $\|Pv\|^2 + \|Qv\|^2 = \|v\|^2$, we get

$$\begin{aligned} \|v(\tau)\| &= \|S(\tau, \tau_0)\, v(\tau_0)\|, \\ &\geqq \|P v(\tau_0)\| - 10^{-2} \|v(\tau_0)\|, \\ &\geqq (1 - 10^{-2} \sqrt{(1 + 10^{-4})}) \, \|P v(\tau_0)\|. \end{aligned}$$

Clearly, if $v(\tau)$ tends to zero, then $Pv(\tau_0) = 0$ and therefore $v(\tau_0) = 0$, which is not the case. This completes the proof of lemma 5. 4.

The following is a direct consequence of the preceding lemmas.

Lemma 5. 5. *If $(x_0, t_0) \in Z$ then there exists a continuous curve $x = \gamma(t)$ in Z with $\gamma(t_0) = x_0$, which is defined for all values of t in $(0, t_0]$.*

Proof. Let C be the set of all continuous curves $x = \gamma(t)$ in Z, defined on some interval $(t_1, t_0]$, with $\gamma(t_0) = x_0$.

We define an ordering on C by saying that $\gamma_1 \leqq \gamma_2$ if the graph of γ_1 is contained in the graph of γ_2. Clearly every chain in C has an upperbound. Since C is nonempty (by lemma 5. 4) it contains a maximal element, $\gamma: (t_1, t_0] \to \mathbb{R}$ (by Zorn's lemma).

If the lower bound of its domain t_1 is positive then $\lim_{t \downarrow t_1} \gamma(t)$ exists by lemma (5. 3) and lemma (5. 4) allows us to extend γ to an interval $(t_1 - \varepsilon, t_0]$, thereby contradicting the maximality of γ.

Thus $t_1 = 0$, and the proof is finished.

We can now prove the main result of this paper. The following is a reformulation of Theorem A of the introduction.

Theorem 5. 6. *For any t_0 the set of singular zeroes $\{(x, t) \in S \mid t = t_0\}$ is discrete.*

Proof. Let $(x_0, t_0) \in S$ be given. We shall construct an interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ which contains at most a finite number of singular zeroes.

For small positive h consider the function w^h defined by

$$w^h(\tau, \xi) = c_h w(x_0, t_0 + h^2(1 - e^{-2\tau}), h e^{-\tau}, \xi)$$

where the constant c_h is chosen in such a way that $w_h(-1, \cdot)$ has L_2-norm one.

Then w^h satisfies

$$(5.\ 3) \qquad w^h_\tau = \frac{1}{2} w^h_{\xi\xi} - \frac{1}{2}(\xi^2 - 1) w^h + Q_h(\xi, \tau) w^h,$$

with $Q_h = 2h^2 e^{-2\tau} q(x + 2he^{-\tau}\xi, t + h^2(1 - e^{-2\tau}))$.

By the compactness result of section 4, we can find a sequence $h_n \downarrow 0$ such that the $w^{h_n}(-1, \cdot)$ converge in E_θ, say to $W(\cdot)$. Using standard results on continuous dependence on initial data and coefficients we see that the corresponding solutions $w^{h_n}(\tau, \xi)$ of (5. 3) converge to the solution of

$$(5.\ 4) \qquad W_\tau = \frac{1}{2} W_{\xi\xi} - \frac{1}{2}(\xi^2 - 1) W$$

with initial value $W(\xi)$.

More precisely, we get convergence in $C([-1, \tau_1); E_\theta)$ for arbitrary finite τ_1 and $\theta < 1$ by applying the variation of constants formula to (5. 3). In particular we get uniform convergence on compact subsets of $[-1, \infty) \times \mathbb{R}$ of w^{h_n} and $w^{h_n}_x$.

The limit solution is real analytic for all $\tau > -1$. Since both u and u_x vanish at (x_0, t_0) all the w^h and w^h_x vanish at $(0, 0)$, so that $(0, 0)$ is a singular zero of the limit solution W.

Since $W(0, \xi)$ is real analytic, $\xi = 0$ is an isolated zero of $W(0, \cdot)$. We can therefore choose an $\varepsilon > 0$ such that $W(0, \pm\varepsilon) \neq 0$. By continuity there also exists a $\delta > 0$ such that $W(\tau, \pm\varepsilon) \neq 0$ for $-\delta \leqq \tau \leqq \delta$. This δ can be chosen in such a way that $W(-\delta, \cdot)$ has only simple zeroes in the interval $-\varepsilon \leqq \xi \leqq \varepsilon$.

Next we choose h so small that w^h also satisfies these conditions, i.e. $w^h(\tau, \pm\varepsilon) \neq 0$ for $-\delta \leqq \tau \leqq \delta$ and simple zeroes only, on the segment $\{-\delta\} \times [-\varepsilon, +\varepsilon]$.

Let N be the set

(5. 5) $$N = \{(x_0 + 2he^{-\tau}\xi, t_0 + h^2(1 - e^{-2\tau})) \mid |\xi| \leqq \varepsilon, |\tau| \leqq \delta\}.$$

Then, by construction, $u \neq 0$ on the sides of N, and u has a finite number of zeroes on the bottom of N.

Suppose there are infinitely many zeroes (x_n, t^*) of u in N, all of which have the same t-coordinate. Then by lemma 5. 5. there exist curves $\gamma_n : (0, t^*] \to \mathbb{R}$ with $\gamma_n(t^*) = x_n$ which lie in Z. The intersections of these curves with the bottom of N produce infinitely many different zeroes of u, since the curves never touch each other (see lemma 5. 3). Clearly this contradicts the way N was constructed.

So we see that for any t, u has only a finite number of zeroes on $N \cap (\{t\} \times \mathbb{R})$, which implies the theorem.

The proof of this theorem allows us to show that zeroes of u "disappear" as time increases.

Let N be as in (5. 5), with ε and δ so small that (x_0, t_0) is the only zero of u in N with $t = t_0$. Then the following holds.

Lemma 5. 7. *$u(x, t)$ has more zeroes on the bottom of N (given by $\tau = -\delta$) than on the top ($\tau = +\delta$).*

Proof. Let w^h and W be as in the proof of the last theorem. By construction the $w^h(0, \xi)$ have exactly one zero in $[-\varepsilon, +\varepsilon]$, namely $\xi = 0$. This zero is a multiple zero since we have assumed $u_x(x_0, t_0) = 0$ all along. Therefore $(0, 0)$ is a multiple zero of W. Analyticity of W allows us to apply the results in [AF], section 5, to conclude that W has at most one zero on $\{\delta\} \times [-\varepsilon, +\varepsilon]$, and at least two zeroes on $\{-\delta\} \times [-\varepsilon, +\varepsilon]$.

Therefore, for small enough ε and δ the same is true of w^h, and hence for u.

Finally we note that theorem B of the introduction can easily be deduced from this last lemma.

Appendix

We recall some results from our paper [A].

In the appendix to that paper we studied the asymptotic behaviour of solutions $u(t)$ of the equation

(A. 1) $$u'(t) + Au(t) = B(t)u(t)$$

where A is a self adjoint operator on a Hilbert space E and $B(t)$ is a bounded family of operators on E. We assumed the following about A:

(1) A is bounded from below.

(2) There is a sequence of intervals $[\alpha_k, \beta_k]$ such that

$$[\alpha_k, \beta_k] \cap \sigma(A) = \emptyset \quad (k, = 1, 2, \ldots)$$

and

$$\alpha_k < \beta_k \leqq \alpha_{k+1} \qquad (k = 1, 2, \ldots).$$

Then, writing $\gamma_k = (\alpha_k + \beta_k)/2$ and $\delta_k = (\beta_k - \alpha_k)/2$ we assumed the following about $B(t)$:

(3) $M = \sup_{t \geqq 0} \|B(t)\|$ is finite, and in fact

$$\sup_{k \geqq 1} \left(\frac{2M}{\delta_k} \right) < 1.$$

Since all the solutions that we dealt with in [A] had to be classical solutions we had to assume that the $B(t)$ were Hölder continuous in the time variable. In the present paper we deal with mild solutions, so that we do not need this extra hypothesis.

The first result from [A], appendix, that is relevant to this paper is lemma 5 (b). This lemma says the following:

Lemma A. 1. *Let $u(t)$ be a mild solution of* (A. 1), *and assume* (1), (2) *and* (3) *hold. Then there is a $\gamma < \infty$ such that*

$$\limsup_{t \to \infty} e^{\gamma t} \|u(t)\| > 0.$$

In other words, solutions of (A. 1) cannot decay faster than exponentially.

The other relevant result from [A] requires an extra hypothesis namely:

$$\lim_{t \to \infty} \|B(t)\| = 0. \tag{4}$$

Let P_k be the spectral projection of the self adjoint operator A, belonging to the interval $(-\infty, \gamma_k]$, and let

$$\hat{u}(t) = u(t)/\|u(t)\|$$

where $u(t)$ is a solution of (A. 1). Then lemma 7 of [A] goes as follows:

Lemma A. 2. *There is a unique integer $k_0 \geqq 1$ such that*

(a) $$\lim_{t \to \infty} \|P_k \hat{u}(t)\| \begin{cases} = 0 & \textit{if} \quad k < k_0, \\ = 1 & \textit{if} \quad k \geqq k_0, \end{cases}$$

(b) $$\limsup_{t \to \infty} e^{\gamma_k t} \|u(t)\| \begin{cases} = 0 & \textit{if} \quad k < k_0, \\ > 1 & \textit{if} \quad k \geqq k_0. \end{cases}$$

In the special case that for any $k \geqq 1$ $\sigma(A) \cap (\beta_k, \alpha_{k+1})$ consists of precisely one simple eigenvalue, whose corresponding eigenfunction will be called φ_k, this lemma implies that

$$\hat{u}(t) \to \pm \varphi_{k_0} \quad \text{as} \quad t \to t_0.$$

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