Non-Linear Diffusion eq. Solutions 5.

Ex. 6.6.14.

$$\begin{cases} \partial_t A = 1 + R\frac{A^2}{H} - A + \partial_x^2 A \\ \partial_t H = Q(A^2 - H) + P\partial_x^2 H \end{cases}$$

With BC: $\partial_x A(t,0) = \partial_x H(t,0) = 0 = \partial_x A(t,L) = \partial_x H(t,L)$.

We look for uniform (so that $\partial_x A \equiv 0 \equiv \partial_x H$) steady states:

$$\begin{cases} 0 = 1 + R\frac{A^2}{H} - A \\ 0 = Q(A^2 - H) \end{cases} \Rightarrow \underline{H = A^2}$$

$$\Rightarrow 1 + R\frac{A^2}{A^2} - A = 0 \Leftrightarrow \underline{\underline{A = 1 + R}}$$

$\therefore$ $\underline{(\bar{A}, \bar{H}) = (1+R, (1+R)^2)}$ is the unique unif. steady state.

When is $(\bar{A}, \bar{H})$ stable?

The Jacobian matrix of $f(A,H) = \left(1 + R\frac{A^2}{H} - A, Q(A^2 - H)\right)$ at $(\bar{A}, \bar{H})$ is:

$$Df(\bar{A}, \bar{H}) = \begin{bmatrix} 2R\frac{A}{H} - 1 & -\frac{RA^2}{H^2} \\ 2QA & -Q \end{bmatrix}\Bigg|_{\substack{\bar{A} = 1+R \\ \bar{H} = (1+R)^2}} = \begin{bmatrix} 2\frac{R}{1+R} - 1 & -\frac{R}{(1+R)^2} \\ 2Q(1+R) & -Q \end{bmatrix} =: M$$

For stability, then we need to have

1) $\underline{\mathrm{Tr}M < 0}$ and 2) $\underline{\mathrm{Det}M > 0}$

... (Why? Since the eigenvalues of M are obtained from

$$\lambda_\pm = \frac{\mathrm{Tr}M \pm \sqrt{(\mathrm{Tr}M)^2 - 4\,\mathrm{Det}M}}{2},$$

so for $\mathrm{Re}\,\lambda_\pm < 0$ we need $\mathrm{Tr}M < 0$ and

$\sqrt{(\mathrm{Tr}M)^2 - 4\,\mathrm{Det}M} < -\mathrm{Tr}M \iff \mathrm{Det}M > 0$.)

The 1)-condition reads

$$\underbrace{2\frac{R}{R+1} - 1}_{\frac{R-1}{R+1}} + (-Q) < 0 \iff \underline{\underline{\frac{R-1}{R+1} < Q}}.$$

The 2)-condition is:

$$\frac{R-1}{R+1}\cdot(-Q) - \left(-\frac{R}{(1+R)^2}\right)\cdot 2Q(1+R) > 0 \quad \| \cdot (1+R)\,Q$$

$$\iff 1 - R + 2R > 0 \iff 1 > -R$$

which is automatically true since R is assumed to be > 0.

Ex. 6.6.15.

As explained in the notes, the stability of the system with diffusion depends on the sign of (see (6.6.6) on p.69)

$$Q := d_1 d_2 (\mu^2)^2 - (d_1 m_{22} + d_2 m_{11})\mu^2 + \mathrm{Det}\, M \, .$$

In our case $d_1 = 1$, $d_2 = P$ and $\mu = \frac{k\pi}{L}$, so the system is still stable iff

$$0 < Q_2 = P\left(\frac{k\pi}{L}\right)^4 - \left(-Q + P\underbrace{\left(\frac{2R}{1+R} - 1\right)}_{R-1/R+1}\right)\left(\frac{k\pi}{L}\right)^2 + 1 + R$$

$$\Leftrightarrow \quad 0 < P\left(\frac{k\pi}{L}\right)^4 + \left(Q - P\frac{R-1}{R+1}\right)\left(\frac{k\pi}{L}\right)^2 + 1 + R \, . \quad (*)$$

If the reverse ineq. $Q_2 < 0$ holds, then, as explained in the notes, the diffusive system is unstable.

ii) Clearly if $P\frac{R-1}{R+1} \le Q$ then $(*)$ is strictly positive. Therefore $\boxed{P\frac{R-1}{R+1} > R}$ is a necessary cond. for "$Q_2 < 0$" and, thus, for instability.

iii) If $P \le 1$, then $P\frac{R-1}{R+1} \le \frac{R-1}{R+1} \overset{\text{by (6.6.17)}}{<} Q$.

$\therefore$ $P > 1$ is a necessary cond. for instability.

Ex. 66.16

$$p(\alpha) := P\alpha^2 + \left\{Q - P\frac{R-1}{R+1}\right\}\alpha + Q$$

Since $P > 0$, the minimum is obtained at

$$0 = p'(\alpha_{min}) = 2P\alpha_{min} + Q - P\frac{R-1}{R+1}$$

$$\Leftrightarrow \alpha_{min} = \frac{P\frac{R-1}{R+1} - Q}{P\,2} = \underline{\frac{1}{2}\left(\frac{R-1}{R+1} - \frac{Q}{P}\right)}.$$

α_{min} is positive if $\underline{P\frac{R-1}{R+1} > Q}$.

Finally

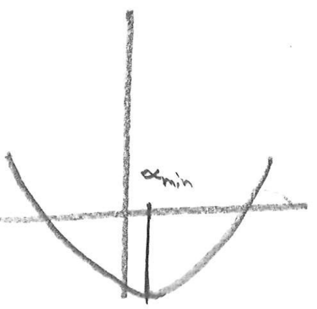
$$p(\alpha_{min}) = \ldots$$

$$= -\frac{\left(Q - P\frac{R-1}{R+1}\right)^2}{4 \cdot P} + Q$$

$$= -\frac{P}{4}\left(\frac{Q}{P} - \frac{R-1}{R+1}\right)^2 + Q$$

$$= -\frac{P}{4}\alpha_{min}^2 + Q$$

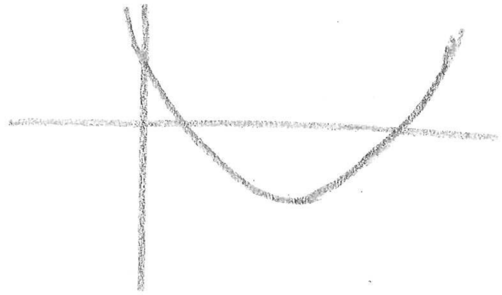
If $\alpha_{min} > 0$ and $p(\alpha_{min}) < 0 \Leftrightarrow \alpha_{min} > 2\sqrt{\frac{Q}{P}}$, the situation looks like:



In our application for "$\alpha = \mu^2$", this is exactly what we are looking for.

Btw. Turn page...

... In our case of $\mathrm{Tr} M < 0$ we know that situation like



where both roots λ_+ and λ_- of $\det(M - \lambda \mathbb{1} - \mu^2 d) < 0$ is impossible.

Ex. 66.17.

Interpretation $\frac{R-1}{R+1} < Q$ would be:
production of the inhibitor H, via $\partial_t H = Q(A^2 - H)$, must be "high enough" w.r.t. production of the activator, for the stability to hold, without the diffusion at least.

Interpretation of the $d_H = P > 1 = d_A$:

If, on the contrary $d_H < d_A$, then the inhibitor would diffuse away slower than the activator, thus "taming down" any local "deviations". This would then preserve the stability of the nondiffusive system.