Exc p. 7.6.

$f: \mathbb{R} \to \mathbb{R}$ is C^1. Let $\varphi \in X = C_0(0,1)$ be *fixed*.

Then, for any $\psi \in X$

$$\| \hat{f}[\varphi + \psi] - \hat{f}[\varphi] - \widehat{Df}[\varphi]\psi \|_\infty$$

$$= \sup_x | \underbrace{f(\varphi(x) + \psi(x)) - f(\varphi(x))}_{\int_0^{\psi(x)} f'(\varphi(x)+z)\,dz} - f'(\varphi(x))\psi(x) |$$

$$\leq \sup_x \int_0^{\psi(x)} | f'(\varphi(x)+z) - f'(\varphi(x)) | \, dz$$

$$\leq \sup_x \int_{-\|\psi\|_\infty}^{\|\psi\|_\infty} | f'(\varphi(x)+z) - f'(\varphi(x)) | \, dz$$

!!!

$$\leq \sup_{|y| \leq \|\varphi\|} \int_{-\|\psi\|_\infty}^{\|\psi\|_\infty} | f'(y+z) - f'(y) | \, dz \cdot \frac{1}{\|\psi\|_\infty} \cdot \|\psi\|_\infty$$

$\to 0$ as $\|\psi\|_\infty \to 0$, for f' is unif. cont. on compacts.

I see no problem in extending to higher derivatives $\hat{D}\hat{f}$, Of course we need to know $f \in C^n$ then.

Ex on p720.

a) $X = C_0(0,1) = Z$, $Y = R$.

b) $\overset{v}{F}(x,y) := \overset{u}{H}(\phi, r) := \phi - G(r\hat{f}(\phi))$

$\Rightarrow D_1F = D_1H(0,r) = 1 - rG$

So, if ϕ_k is kth eigenvector of G,

$D_1H(0,r)\phi_k = \phi_k - r\phi_k = \underline{\left\{1 - \frac{r}{(2\pi k)^2}\right\}}\phi_k$

$\therefore$ If $r \neq (2\pi k)^2$ for all $k \in \mathbb{N}$ then
$R(D_1H(0,r)) = X$.

c) If $r_k = (2\pi k)^2$, then
$N_k = \operatorname{span}\phi_k$ and $R_k = \operatorname{span}\{\phi_j : j \neq k\}$
$\Rightarrow X = N_k \oplus R_k$.

d) $X_0^k = \{\varphi \in X : \langle\phi_k|\varphi\rangle = 0\}$, $Z_0^k = \operatorname{span}\phi_k$

e) $P = |\phi_k\rangle\langle\phi_k|$, $Q = \sum_{n \neq k} |\phi_n\rangle\langle\phi_n|$

f) $D_{21}H(0,r_k) = -G$, so if $v_0^k := \phi_k$

$-G\phi_k = -\frac{r_k}{(2\pi k)^2}\phi_k \in N_k \subseteq R_k^{\perp}$.

Ex. on p. 7.21

a) So we pick $r_k = (\pi k)^2$, $k \geq 2$ where we have bifurcation from the trivial solution $\bar{0}$. The bifurcation, parameterized by $\psi(s)$, "happens in the direction ϕ_k" in the sense (see notes p. 7.18.)

$$\frac{d}{ds}\Big|_{s=0} \psi(s) = \phi_k$$

Then $\psi(s,x) = \underbrace{\psi(0,x)}_{=0} + s\,\phi_k(x) + s\,\varepsilon(s,x)$,

where $\varepsilon(s) \to 0$ as $s \to 0$.

$\Rightarrow \psi(s) = s\,(\phi_k + \varepsilon(s))$.

So pick s small enough so that $|\varepsilon(s)| < \frac{1}{2}\|\phi_k\|$ and you can be sure there will be both positive and neg. values.

b) If $k = 1$ $\phi_1(x) = \sin(\pi x) \approx \pi x$ for small x.

Thus $\psi(s,x) \approx s(x\pi + s\,\varepsilon(s,x)) < 0 \Leftrightarrow \varepsilon(s,x) < -\frac{x\pi}{s}$

So, it would seem possible if

$$\partial_x^2 \psi(s,0) = \varepsilon(s,0) < 0 = \lim_{x \to 0} -\frac{x\pi}{s}$$

for some s "pretty small". This is no proof, simply the idea.

2nd. Ex. on p. 7.21

By def. of $\hat{y}(s)$ we have $\Phi(s, \hat{y}(s)) = 0, \ \forall s$.

$$\Rightarrow 0 = \frac{d}{ds}\underbrace{\Phi(s, \hat{y}(s))}_{\int_0^1 dt\, \underbrace{D_1\Phi(st\hat{v}_0, \hat{y}(s))\hat{v}_0}_{\langle z_0^*, D_1F(st\hat{v}_0, \hat{y}(s))\hat{v}_0\rangle z_0}}$$

$$\Leftrightarrow 0 = \int_0^1 dt \langle z_0^*, \underbrace{\frac{d}{ds} D_1F(st\hat{v}_0, \hat{y}(s))\hat{v}_0}\rangle z_0$$

$$t\, D_{11}F(st\hat{v}_0, \hat{y}(s))[\hat{v}_0, \hat{v}_0] + D_{21}F(-\|-)\dot{\hat{y}}(s)$$

$$= \int_0^1 dt\, t \langle z_0^*, D_{11}F(st\hat{v}_0, \hat{y}(s))[\hat{v}_0, \hat{v}_0]\rangle z_0 + \langle z_0^*, D_{21}F(st\hat{v}_0, \hat{y}(s))\hat{v}_0\rangle z_0\, \dot{\hat{y}}(s)$$

So, at $s = 0$ we get, as we perform the trivial integral

$$0 = \frac{1}{2}\langle z_0^*, D_{11}F(0, y_0)[\hat{v}_0, \hat{v}_0]\rangle z_0 + \langle z_0^*, D_{21}F(0, y_0)\hat{v}_0\rangle z_0\, \dot{\hat{y}}(0)$$

from which we get, since $\langle z_0^*, \underbrace{D_{21}F(0, y_0)\hat{v}_0\rangle}_{\in \mathbb{R}} \neq 0$

$$\left.\frac{d}{ds}\right|_{s=0} \hat{y}(s) = -\frac{1}{2}\frac{\langle z_0^*, D_{11}F(0, y_0)[\hat{v}_0, \hat{v}_0]\rangle}{\langle z_0^*, D_{21}F(0, y_0)\hat{v}_0\rangle} \qquad \square$$

Ex. on p. 7.23

To apply the previous formula, one must calculate $D_{21}H(0,r)$ & $D_{11}H(0,r)$. Recall that

$$D_1 H(\phi,r) = \mathbb{1} - r\mathcal{G}\hat{f'}(\phi) \quad , \text{ meaning}$$

$$D_1 H(\phi,r)\psi = \psi - r\mathcal{G}[\underbrace{f'\circ\phi}_{\text{multiplication operator.}}\cdot\psi] \xrightarrow{\phi\mapsto\bar{0}} \underline{\underline{\psi - r\overset{\overset{1}{\shortparallel}}{f'(0)}\,\mathcal{G}[\psi]}} .$$

Now, $\underline{D_{21}H(0,r)\psi = -\mathcal{G}[\psi]}$. Finally

$$D_{11}H(\phi,r)(\psi,\psi) = \underline{-r\mathcal{G}[(D_{\phi\phi}\hat{f})[\phi](\psi,\psi)]} \text{ , where}$$

$$(D_{\phi\phi}\hat{f})[\phi](\psi,\psi) = D_\phi[\hat{f'}(\phi)\psi](\psi)$$
$$= \underline{\underline{\hat{f''}[\phi]\cdot\psi\cdot\psi}} ,$$

The very same principle as before, now with function $f'(\phi)\cdot\psi$.

Next, we have $\hat{v}_0 = \phi_k = \sqrt{2}\sin(k\pi\,\cdot)$, $z_0 = \phi_k$ too, and $\langle\cdot,\cdot\rangle = (\cdot,\cdot)_{L^2[0,1]}$. Also $r_k = (k\pi)^2$

Now, since ϕ_k is an eigenvector of $\mathcal{G}$ with e-value $\frac{1}{(k\pi)^2}$,

$$D\hat{y}(0) = -\frac{1}{2}\,\frac{(\phi_k, -r_k f''(0)\,\mathcal{G}\phi_k^2)}{(\phi_k, -\mathcal{G}\phi_k)} = -\frac{r_k f''(0)}{2}\cdot\frac{(\phi_k,\mathcal{G}\phi_k^2)}{\frac{1}{(k\pi)^2}\|\phi_k\|^2} .$$

Finally, note that $(\phi_k,\mathcal{G}\phi_k^2) = (\mathcal{G}\phi_k,\phi_k^2) = \frac{1}{(k\pi)^2}(\phi_k,\phi_k^2)$ so

$$D\hat{y}(0) = -\frac{1}{2}(k\pi)^2 f''(0)\underbrace{(\phi_k,\phi_k^2)}_{\int_0^1(\sqrt{2}\sin(k\pi x))^3dx}$$

Done.

1-st on p 7.26

By def. of $\tilde{w}$ and $\tilde{\lambda}$:

$$\underbrace{F(a, y, \tilde{w}(a,y), \tilde{\lambda}(a,y))}_{} = 0, \quad \forall y.$$

$$D_1F(a,y)(\hat{v}_0 + \tilde{w}(a,y)) - \lambda(a,y)(\hat{v}_0 + \tilde{w}(a,y))$$

So, applying $\frac{d}{dy}$ on both sides gives us

$$0 = D_{21}F(a,y)(\hat{v}_0 + \tilde{w}) + D_1F(a,y)\,\partial_y\tilde{w}(a,y) - \partial_y\lambda(a,y)(\hat{v}_0 + \tilde{w}) - \lambda(a,y)\,\partial_y\tilde{w}(a,y).$$

Evaluating at $y = y_0$ (recall $\tilde{w}(a,y_0) = 0$)

$$0 = D_{21}F(a,y_0)\,\hat{v}_0 + D_1F(a,y_0)\,\partial_y\tilde{w}(a,y_0) - \partial_y\lambda(a,y_0)\,\hat{v}_0 - \lambda_0\,\partial_y\tilde{w}(a,y_0)$$

$$= \left[D_{21}F(a,y_0) - \partial_y\lambda(a,y_0)\right]\hat{v}_0 + \left[D_1F(a,y_0) - \lambda_0\right]\partial_y\tilde{w}(a,y_0)$$

Applying $\langle z_0^*, \cdot\rangle$ to both sides:

$$0 = \langle z_0^*, D_{21}F(a,y_0)\hat{v}_0\rangle - \partial_y\lambda(a,y_0) + \overbrace{\langle z_0^*, \underbrace{[D_1F(a,y_0) - \lambda_0]\,\partial_y\tilde{w}(a,y_0)}_{\in\,\mathrm{span}\{\hat{v}_0\}^\perp}\rangle}^{=0}$$

$$\therefore \quad \partial_y\lambda(a,y_0) = \lambda_*'(y_0) = \langle z_0^*, D_{21}F(a,y_0)\hat{v}_0\rangle.$$

2nd. Ex. on p. 7.26

Well, if $D_{21}F(0,y_0)\hat{v}_0 \notin R$, then

$$\langle z_0^*, D_{21}F(0,y_0)\hat{v}_0\rangle \neq 0,$$

which by prev. ex. means $\dot{\lambda}_+(y_0) \neq 0$.

3rd. —"— 7.2.6

Precisely the same principle as in exercises just above:
First differentiate the defining eq. of the non-trivial-branch-eigenvalue $F_e(x(s), y(s), w(x(s), y(s)), \lambda(x(s), y(s))) = 0$
w.r.t. s.

Secondly, apply $\langle z_0^*,$ to the resulting eq. to eliminate terms which have values in R.

Done. (Ask the details from me if you really need to see them. They won't make you wiser I think.)

4th. —"— By 3rd. ex.

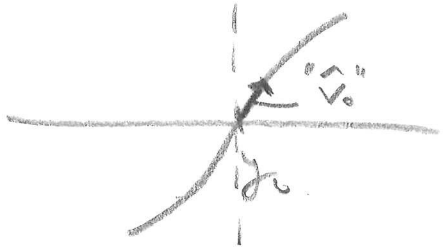
$$\dot{\lambda}_n(0) = \underbrace{\dot{\lambda}_+(y_0)}_{\langle z_0^*, D_{21}F(0,y_0)v_0\rangle}\ \underbrace{D\hat{y}(0)}_{-\frac{1}{2}\frac{\langle z_0^*, D_{11}F(0,y_0)[v_0,v_0]\rangle}{\langle z_0^*, D_{21}F(0,y_0)v_0\rangle}} + \langle z_0^*, D_{11}F(0,y_0)[v_0,v_0]\rangle$$

$$= \frac{1}{2}\langle z_0^*, D_{11}F(0,y_0)[v_0,v_0]\rangle = \overbrace{-\frac{1}{2}\frac{\langle z_0^*, D_{11}F(0,y_0)[v_0,v_0]\rangle}{\dot{\lambda}_+(y_0)}}^{D\hat{y}(0)} \cdot \dot{\lambda}_+(y_0)$$

Done.

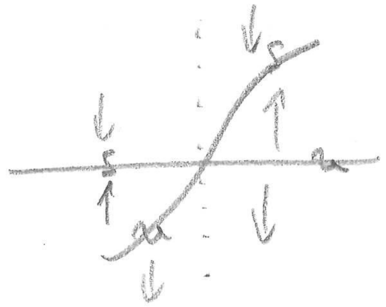
5 ch – " – 7.26

If $D\hat{y}(0) > 0$ the situation looks like this:



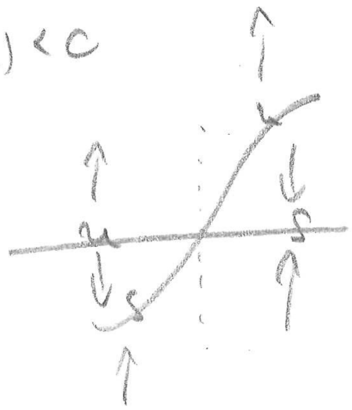
In which case the stabilities can look like

$\dot{\lambda}_*(y_0) > 0$

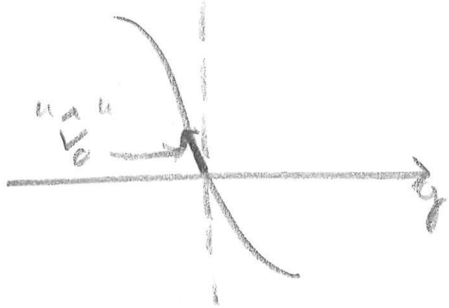


or

$\dot{\lambda}_*(y_0) < 0$

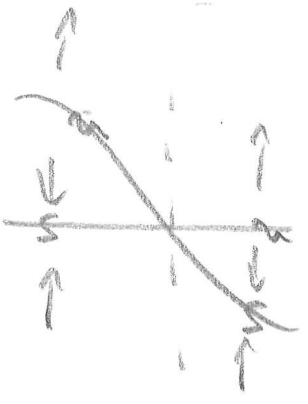


If $D\hat{y}(0) < 0$ the situation looks like:



———— " ———— stabilities can look like:

$\dot{\lambda}_*(y_0) > 0$



or

$\dot{\lambda}_*(y_0) < 0$