Non-Linear Diffusion Problems, Solutions 2.

5.2.6 $\quad \partial_t u(t,x,y) = d\Delta u(t,x,y) + r u(t,x,y),$

where $x \in [0, L_x]$ and $y \in [0, L_y]$. We pose boundary cond.

$$\partial_x u(t,0,y) = 0 = \partial_x u(t,L_x,y) \; ; \; \partial_y u(t,x,0) = 0 = \partial_y u(t,x,L_y).$$

Let's try separation of variables: Plug the "ansatz"

$u(t,x,y) := a(t)\, b(x)\, c(y)$

into the diffusion eq. to get

$$\dot{a}(t)\, b(x)\, c(y) = d\big(a(t)\, b''(x)\, c(y) + a(t)\, b(x)\, c''(y)\big) + r\, a(t) b(x) c(y)$$

Divide by $a \cdot b \cdot c$ to obtain

$$\frac{\dot{a}(t)}{a(t)} = d\left(\frac{b''(x)}{b(x)} + \frac{c''(y)}{c(y)}\right) + r, \quad (\forall t, x, y). \qquad (\dagger)$$

This can only be true if there are constants λ, C_x, C_y such that

$$\frac{\dot{a}(t)}{a(t)} = \lambda \;, \quad \frac{b''(x)}{b(x)} = -C_x \;, \quad \frac{c''(y)}{c(y)} = -C_y \;,$$

and by $(\dagger)$ these also have to satisfy $\lambda = -d(C_x + C_y) + r$.

There will be further restrictions from the <u>boundary cond.</u>!

... Consider, for example,

$$b''(x) = -C_x\, b(x) .$$

The solution is, generally, a combination

$$b(x) = A\cos(\sqrt{C_x}\,x) + B\sin(\sqrt{C_x}\,x) .$$

However, "$\partial_x n(t,0,y)=0$" implies <u>B must be zero.</u>
Also, "$\partial_x n(t,L,y)=0$" implies

$$-\sin(\sqrt{C_x}\,L) = 0 \iff \sqrt{C_x}\cdot L = k_x\pi \quad , k_x = 0,1,2\ldots$$

$$\iff \underline{C_x = \left(\frac{k_x\pi}{L_x}\right)^2} .$$

$\Rightarrow$ The possible solutions can be numbered

$$b_{k_x}(x) = \cos\left(\frac{k_x\pi}{L_x}x\right) \quad , \quad k_x = 0,1,2\ldots$$

Same thing for $c(y)$, only different L_y:

$$c_{k_y}(y) = \cos\left(\frac{k_y\pi}{L_y}y\right) \quad , k_y = 0,1,2\ldots$$

Once k_x, k_y are chosen, $\lambda(k_x,k_y)$ is <u>determined:</u>

$$\underline{\lambda(k_x,k_y) = r - d\left\{\left(\frac{k_x\pi}{L_x}\right)^2 + \left(\frac{k_y\pi}{L_y}\right)^2\right\}} \quad (*)$$

... Put it all together and get, for each pair (k_x, k_y),

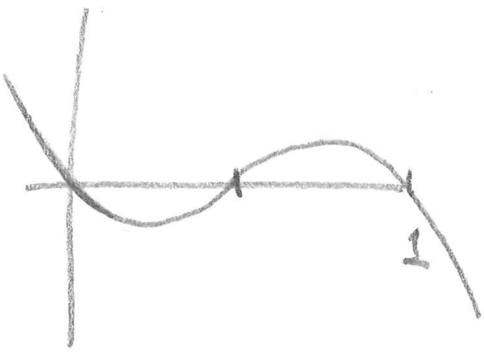
$$u_{k_x,k_y}(t,x,y) = e^{\lambda(k_x,k_y)t} \cdot \cos\left(\frac{k_x\pi}{L_x}x\right)\cos\left(\frac{k_y\pi}{L_y}y\right) .$$

It is curious that the growth rate λ is determined by the ratios $\frac{k_x}{L_x}$ and $\frac{k_y}{L_y}$. In particular, it can not get "just any values".

The degeneracy of a particular λ is the number of pairs $(k_x, k_y) \in \mathbb{N}^2$ that solve ⓧ.

Thus, it can be seen that the density and degeneracy of possible λ:s increases as L_x or L_y does.

② $f(u) := u(u - \frac{1}{2})(1-u)$



$$\begin{cases} u'(x) = v(x) \\ v'(x) = -f(u(x)) \end{cases}$$

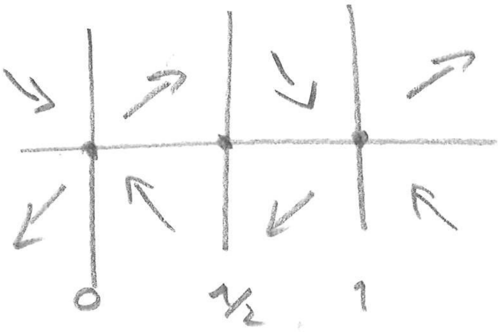
We have Hamiltonian from

$u' = \partial_v H(u,v) \Rightarrow H = \frac{1}{2}v^2 + \alpha(u)$

$v' = -\partial_u H(u,v) \Rightarrow H = -\int_0^u f(w)\,dw + \beta(v)$

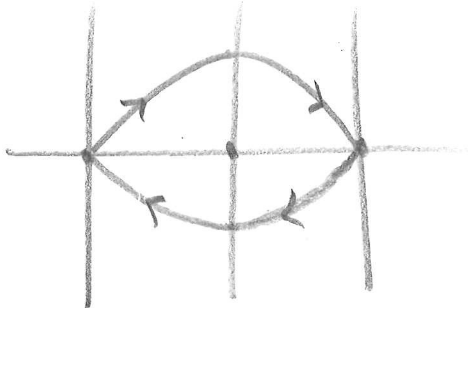
$\therefore H(u,v) = \frac{1}{2}v^2 + F(u)$, $F(u) = -\int_0^u f$, is a constant of motion.

i) Phase portrait: $u' = v \wedge v' = -f(u)$ tell us that the general flow pattern looks like

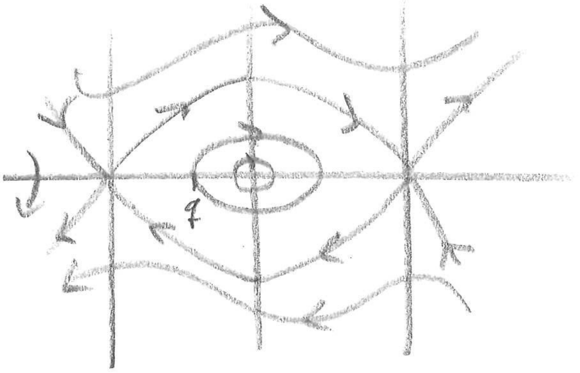


To be able to "connect the orbits" we must find the level sets of H.

Eg. Since $H(0,0) = 0 = H(1,0)$, and $H(u,0) < 0$, $\forall u \neq 0 \vee 1$, and considering the flow pattern we know that point $(0,0)$ connects to $(1,0)$ as follows:



Similarly, rest of the orbits don't have a choice but to look like:



ii) To get solutions to the No-flux (BVP)(r), we change $f \mapsto rf$, and look for orbits that start from the u-axis (so that $\phi'(0) = v_0 = 0$) and end up, after time 1, to u-axis (so that $\phi'(1) = v(1) = 0$). ϕ corresponds to u.

So we need the orbit time. It's obtained from this:

$E(q,0) = E(u,v)$ for all u,v on the orbit starting from $(q,0)$

$\Leftrightarrow F_r(q) = \frac{1}{2}v^2 + F_r(u) \Leftrightarrow v = \frac{du}{dx} = \sqrt{2(F_r(q) - F_r(u))}$

$\Rightarrow x_q(u) = \displaystyle\int_q^u \frac{dw}{\sqrt{2(F_r(q) - F_r(w))}}$

Let $T_r(q)$ denote the time the orbit hits the u-axis:

$T_r(q) = \displaystyle\int_q^{q_+} \frac{dw}{\sqrt{2(F_r(q) - F_r(w))}}$, $q \in [0, \frac{1}{2}]$

Idea: To find the right orbits, for a fixed r, we must look for q such that either, $T_r(q) = 1$, or, $nT_r(q) = 1$ for some $n = 2, 3 \ldots$. The latter correspond to orbits that travel ...

... more than half a round. The sufficient condition to have ANY suitable orbits of n half turns is

$\inf_{q} n T_r(q) < 1$.

Now, if T_r is monotone, then we find $\inf T_r$ as the limit

$$\lim_{q \to 1/2} T_r(q) = \frac{\pi}{\sqrt{r f'(\frac{1}{2})}} = \frac{2\pi}{\sqrt{r}}$$

as shown in lectures.

I'm going to "prove by hand waving" that T_r is monotone. Consider this:

$T_r(q_2) < T_r(q_1)$ for $q_2 < q_1$ would mean that the path travels a longer route, starting from q_2, in less time. By $u' = v$, the only way such a thing may happen is to "go high enough fast enough" (the flow to the right is stronger high up) So how fast must one go up ???

Well, consider $f(u) = -Cu$ linear. Then the solutions are all oscillating with same frequency, irrespective of q. Thus the linear case f seems to be the limiting situation. So, what does weaker than linear mean here? It means concave on $(0, \frac{1}{2})$ and convex on $(\frac{1}{2}, 1)$. It requires a bit of meditation but it's clear that this will suffice. And, our f fulfills it. $\Rightarrow$ Monotone. "QED"

So, to put it together, if

$$\frac{n\,2\pi}{\sqrt{r}} < 1 \iff r > \sqrt{2\pi n}$$

then there will be at least $2 \cdot n$ suitable orbits (2 comes from the choice of starting below or above $u = \frac{1}{2}$).

iii) As seen from the phase portrait, orbits starting from v axis will never return. So no solutions excepting the trivial $u \equiv 0$.

iv) They are the periodic orbits plus the connecting orbits between $(0,0)$ and $(1,0)$.

(3) i) Assume ϕ is a solution to (BVP)(r):

$$\begin{cases} \phi'' + r f(\phi) = 0 & (0 < x < 1) \\ \phi'(0) = 0 = \phi'(1) & , \end{cases}$$

and define

$$\tilde{\phi}(x) := \phi(1-x).$$

Then $\partial_x^2 \tilde{\phi}(x) = \partial_x^2 \phi(1-x) = (-1)^2 \phi_{xx}(1-x) = \tilde{\phi}_{xx}(x)$

and thus $\tilde{\phi}$, clearly, is also a solution to (BVP)(r).

ii) It should be clear enough both extended functions, ϕ and $\tilde{\phi}$, satisfy $\phi'' + r f(\phi)$ on $\mathbb{R} \setminus \mathbb{Z}$. Then the thing to note is that the no-flux boundary cond. ensures the extensions will be differentiable once. Actually, it's even twice diff.! For example around 0: $\forall \varepsilon > 0$

$$\partial_x^2 \phi(-\varepsilon) = \phi''(-\varepsilon) = -r f(\underbrace{\phi(-\varepsilon)}_{\phi(\varepsilon) \ (*)}) = -r f(\phi(\varepsilon)) = \phi''(\varepsilon)$$

$\rightarrow \phi''$ is cont. at 0.

((*) It follows from our strategy of extension that always $\phi(-x) = \phi(x)$.)

Next, ϕ is symmetric iff, by def., $\tilde{\phi}(x) = \phi(x)$ $\forall x$.

That's the same as $\phi(1-x) = \phi(x)$. But, by 2-periodicity of our extension, $\phi(1-x) = \phi(-(-1+x)) = \phi(1+x)$, which says ϕ is 1-periodic.

iii) $\phi_k(x) := \phi(kx)$ where $k \in \mathbb{N}$.

Now $\partial_x^2 \phi_k(x) = \partial_x^2 [\phi(kx)] = k^2 \phi''(kx)$. Thus

$$\partial_x^2 \phi_k(x) + k^2 r\, f(\phi_k(x)) = k^2 \{ \underbrace{\phi''(kx) + r f(\phi(kx))}_{=0} \} = 0.$$

$\therefore$ ϕ_k satisfies (BVP)($k^2 r$).

Let $k = 2n$.

$\Rightarrow \phi_k(x+1) = \phi(2nx + 2n) = \phi(2nx) = \phi_k(x)$. Done.

iv) $\phi_{1/2}(x) := \phi(\frac{1}{2}x)$. As before, it's easily seen that

$\partial_x^2 \phi_{1/2} + \frac{r}{4} f(\phi_{1/2}) = 0$ on $\mathbb{R}$.

For $\phi_{1/2}$ to satisfy the BC ϕ must be symmetric, well actually we only need $\phi'(\frac{1}{2}) = 2\phi_{1/2}'(1) = 0$, which certainly follows if $\phi'(x) = \phi'(1-x)$ for $x \in [0,1]$.

For each cont. func. f there is a smallest period, unless it's constant. The most elementary way to see this is to assume, on the contrary, that for each δ, there is $\sigma < \delta$ such that $f(x+\sigma) = f(x)$. Now the cont. periodic func. like f will attain both its max., say at x_1, and its min., at x_2. And the clue: By σ periodicity, these max. and min. are also attained on $x_1 + n\sigma$, $x_2 + m\sigma$, $n, m \in \mathbb{Z}$! Clearly f cannot be cont. unless it's constant. (Exercise !!).

v) Because ... our extension tricks only depended on the $x \in [0,1]$ or $\mathbb{R}$. Thus we can still do the same $\tilde{f}(x) = f(1-x)$ ∧ $f(x+2n) := f(x)$ ect.

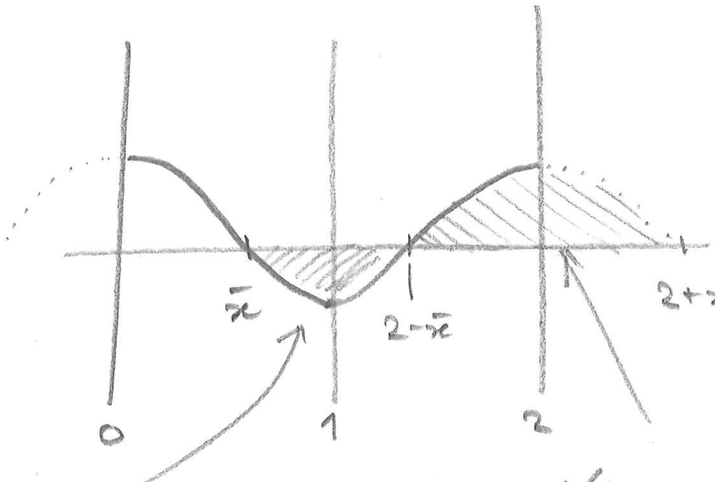
vi) Sc, assume $f(\bar{x}) = 0$, $\bar{x} \in [0,1]$. Then
$f(2-\bar{x}) = f(-\bar{x}) = f(\bar{x}) = 0$ too.

Now define $\varphi_1(x) := f(x+\bar{x})$, $\varphi_2(x) := f(x+2-\bar{x})$.

Then $\varphi_1(0) = 0 = \varphi_1(2-2\bar{x})$

and $\varphi_2(0) = 0 = \varphi_2(2\bar{x})$

<u>Picture:</u>



this one corresponds to φ_2

this corresponds to φ_1.

$\Rightarrow \varphi_1$ is solution to $\underset{\text{Big mons.}}{(BVP)}$ $(L = 2-2\bar{x})$ while

φ_2 is solution to $\underset{\text{Big m.}}{(BVP)}$ $(L = 2\bar{x})$.

These L can be "<u>adjusted to 1</u>" at the cost of changing r.