## 5.3.1

We are given the fundamental sol.

$$u(t,x) = \frac{1}{4\pi d t} e^{-|x|^2/4dt}.$$

**(i) and (ii):** If we're looking for the moving front where the "shift from 0 to ∞" takes place, then we should concentrate on the exponential term.

To get an upper estimate, it's quite natural to demand

$$e^{rt - x(t)^2/4\pi d} \leq e^{-\varepsilon t},$$

for then, surely $u(t, x(t)) \to 0$ as $t \to \infty$. That cond. translates to

$$rt - \frac{x(t)^2}{4dt} \leq -\varepsilon t \iff x(t)^2 \geq 4d(r+\varepsilon)t^2.$$

(that's practically the same cond. as in the notes.)

Similarly, to obtain a lower estimate for the speed, we demand

$$e^{rt - x(t)^2/4dt} \geq e^{\varepsilon t},$$

for then $u(t, x(t)) \to \infty$, as $t \to \infty$.

$$\Rightarrow rt - \frac{x(t)^2}{4dt} \geq \varepsilon t \iff x(t)^2 \leq 4d(r-\varepsilon)t^2.$$

... (iii) So the action is going on somewhere in

$$\sqrt{4d(r-\varepsilon)}\,t \le \text{"Action"} \le \sqrt{4d(r+\varepsilon)}\,t$$

Thus the speed $\frac{x(t)}{t}$ of that front is between $\sqrt{4d(r-\varepsilon)}$ and $\sqrt{4d(r+\varepsilon)}$. Thus, roughly $2\sqrt{dr}$.

(Note still, that the width of the front, $4\sqrt{d\varepsilon}\,t$, is also increasing. This can be "collected" by starting from $e^{rt-x^2/4dt} \le e^{-\varepsilon\sqrt{t}}$ for example.)

(iv) Let $u(t,x) := w(x\cdot\hat{v} - ct)$, where $\|\hat{v}\| = 1$.

Then

$\partial_t u = -c w'$

$\partial_{x_1} u = v_1 w'$, $\partial_{x_2} u = v_2 w'$ $\leadsto$ $\nabla u = \hat{v} w'$

$\Delta u = \hat{v}\cdot\hat{v}\, w'' = w''$.

Thus $\partial_t u = d\Delta u + ru$ becomes

$-cw' = dw'' + rw$.

(v) Try $w(\xi) := e^{\lambda\xi}$ to obtain the characteristic eq.:

$$-c\lambda w = d\lambda^2 w + rw \iff d\lambda^2 + c\lambda + r = 0$$

$$\iff \lambda = \frac{-c \pm \sqrt{c^2 - 4dr}}{2d}$$

(vi) We do not want $\lambda$ to be complex, because that would mean oscillating solution and thus negative values. Thus the minimum speed is

$\underline{c_m = 2\sqrt{dr}}$.

The corresponding $\lambda_m$ becomes $-\frac{\sqrt{r}}{\sqrt{d}}$.

(vii) Well, the diffusion speed, $2\sqrt{rd}$, sets the minimum, just as ~~delay~~ does the delay between rockets igniting each other. You can, "of course", make waves that go faster. (In the extreme case, set all exploding at once.)

(viii) Plugging $\lambda_m$ into $w(\xi) = e^{\lambda_m \xi}$, we get for $u$

$$u_m(t, x) = e^{-\sqrt{\frac{r}{d}}(x \cdot \hat{v} - 2\sqrt{dr}\, t)} = e^{-\sqrt{\frac{r}{d}}\, x \cdot \hat{v}} \cdot \underline{\underline{e^{2rt}}}$$

$\therefore$ For fixed $x$ the growth rate is $\underline{e^{2rt}}$.

Compare for uniform population, obtained from $\partial_t u(t,x) = r u(t,x)$ $(\partial_x^2 u \equiv 0)$, $\underline{u(t,x) = e^{rt} u_0}$.