E.1

Exercises for 14 November

1. Exercise 5.2.6 of syllabus_ch5 (page 47)
2. Consider $f(u) = u(u-\frac{1}{2})(1-u)$. Note that ~~f(u)~~ $f(1-u) = -f(u)$, i.e., f is symmetric with respect to reflection in $u = \frac{1}{2}$.
   i) Draw the phase portrait for $\begin{cases} u' = v \\ v' = -f(u) \end{cases}$
   ii) Use the information provided by the phaseportrait and "time" considerations to classify solutions of
   $$(BVP)_{no\text{-}flux} \begin{cases} \phi'' + r f(\phi) = 0, 0<x<1 \\ \phi'(0) = 0 = \phi'(1) \end{cases}$$
   iii) What can you say about solutions of $(BVP)_{big\text{-}monster}$?
   iv) What can you say about bounded (and positive) solutions of $\phi'' + r f(\phi) = 0$ on $-\infty < x < +\infty$?

3. Creating new solutions from a known solution by symmetric extension and scaling of the one-dimensional spatial variable x:

We consider $(BVP)_{no\text{-}flux}$ with parameter r, i.e.,

$$\phi'' + r f(\phi) = 0 , \quad 0 < x < 1$$
$$\phi'(0) = 0 = \phi'(1)$$

$f: \mathbb{R} \to \mathbb{R}$ is at least continuous

E.2 To indicate the dependence on r, we shall write (BVP)(r) and, unless stated explicitly otherwise, restrict ~~to~~ to no-flux boundary conditions on $[0,1]$.

Our starting point is that we know, in one way or another, a solution ϕ of (BVP)(r).

i) Define
$$\tilde{\phi}(x) = \phi(1-x)$$

Verify that $\tilde{\phi}$ is a solution of (BVP)(r)

<u>Definition</u> We call ϕ <u>symmetric</u> if $\tilde{\phi} = \phi$

We extend both ϕ and $\tilde{\phi}$ to functions defined on $\mathbb{R}$ by reflection in $x=1$ and next periodic continuation with period 2, i.e., we define

$$\phi(1+x) = \phi(1-x) \quad , \quad 0 \leq x \leq 1$$

$$\phi(x) = \phi(x \bmod 2) \quad , \quad x \in \mathbb{R}$$

and similarly for $\tilde{\phi}$.

ii) Check that ϕ and $\tilde{\phi}$ satisfy the (second order) differential equation for all $x \in \mathbb{R}$

Check that $\tilde{\phi}(x) = \phi(1+x)$

Check that ϕ is symmetric iff the extended function ϕ has period 1.

E.3

iii) Assume that ϕ is not symmetric. Define for $k \in \mathbb{N}$

$$\phi_k(x) = \phi(kx)$$

Show that ϕ_k is a solution of BVP$(k^2 r)$.

Show that ϕ_k has period 1 if k is even.

iv) Now assume that, to the contrary, ϕ is symmetric.
Define

$$\phi_{1/2}(x) = \phi(\tfrac{1}{2}x)$$

Verify that $\phi_{1/2}$ is a solution of (BVP)$(\frac{r}{4})$

Note that, repeating this procedure if $\phi_{1/2}$ is itself symmetric, we ultimately arrive at a solution that is not symmetric, but for a different value of r.

v) Consider the system of equations

$$d\phi'' + r f(\phi) = 0 \quad , \quad 0 < x < 1$$

$$\phi'(0) = 0 = \phi'(1)$$

where $f: \mathbb{R}^m \to \mathbb{R}^m$, $\phi(x) \in \mathbb{R}^m$, $d = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{pmatrix}$

Provide an argument that "shows" that all the above relationships can be extended to this situation.

vi) In the context of a single equation, assume that there exists a unique zero $\bar{x}$ of ϕ in $[0,1]$. Verify that $\phi(2-\bar{x}) = 0$. Verify that $\phi(x+\bar{x}) = 0$ for $x = 0$ and for $x = 2 - 2\bar{x}$ and that $\phi(x + 2 - \bar{x}) = 0$ for $x = 0$ and $x = 2\bar{x}$. So if $\bar{x} = \frac{1}{2}$ both are solutions of (BVP)$_{\text{big monster}}(r)$, but in general one has to adjust r for each separately to obtain a solution of (BVP) big monster