Dirichlet problem at infinity for the minimal graph equation on Cartan-Hadamard manifolds

Ilkka Holopainen

University of Helsinki

ilkka.holopainen@helsinki.fi

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based on joint works with
Jean-Baptiste Casteras, Esko Heinonen, and Jaime Ripoll

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Minimal graph equation

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u\colon \bar{\Omega} \to \mathbb{R}$ a smooth function.
Let

$$S_u = \left\{ \left(x, u(x)\right) \colon x \in \Omega \right\} \subset \Omega \times \mathbb{R}$$

be the graph of u.
The area (n-dim. measure) of S_u is:

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx.$$

We want to minimize (the area functional) $\mathcal{A}$ among all functions with boundary values $u|\partial\Omega$.
Let $\eta \in C_0^\infty(\Omega)$ and consider $u + t\eta,\ t \in \mathbb{R}$.

Minimal graph equation

If u is a minimizer, then

$$0 = \frac{d}{dt}\mathcal{A}(u + t\eta)_{|t=0} = \cdots = \int_{\Omega} \frac{\langle \nabla u, \nabla \eta \rangle dx}{\sqrt{1 + |\nabla u|^2}}.$$

This is the *weak form* of the *minimal graph equation*

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0.$$

(The left hand side is also called the *mean curvature operator*.)

Connection to harmonic functions/mappings

Intuitively: Suppose $S_u \approx$ "flat", i.e. $|\nabla u| \approx 0$. Then

$$\int_\Omega \sqrt{1 + |\nabla u|^2} dx = \int_\Omega \left(1 + \tfrac{1}{2}|\nabla u|^2 + \text{ higher order terms}\right) dx$$
$$\approx m(\Omega) + \tfrac{1}{2} \int_\Omega |\nabla u|^2 dx.$$

Minimizers of

$$\int_\Omega |\nabla u|^2 dx$$

are *harmonic functions*.

Connection to harmonic functions/mappings

Precisely: Let $N^n \subset \mathbb{R}^m$ be a (smooth) submanifold and

$$\pi\colon N^n \hookrightarrow M^m, \quad \pi = (\pi_1, \pi_2, \ldots, \pi_m)$$

the inclusion map. Then

$$\Delta^N \pi = (\Delta^N \pi_1, \Delta^N \pi_2, \ldots, \Delta^N \pi_m) = H,$$

where Δ^N is the Laplace-Beltrami operator on N (w.r.t. induced structure) and H is the mean curvature (vector).
In particular, if N is a minimal submanifold ($\iff$ $H \equiv 0$), then each π_i is a harmonic function on N.

Connection to harmonic functions/mappings

For minimal graphs $S_u \subset \mathbb{R}^{n+1}$ this means:

- $$\pi_{\mathbb{R}} \colon S_u \to \mathbb{R}, \quad \pi_{\mathbb{R}}\big(x, u(x)\big) = u(x) \quad \text{(height function)}$$
  is a harmonic function on S_u.
- $$\pi_{\mathbb{R}^n} \colon S_u \to \mathbb{R}^n, \quad \pi_{\mathbb{R}^n}\big(x, u(x)\big) = x$$
  is a harmonic mapping on S_u.

Setting

From now on:
M: n-dimensional ($n \geq 2$) *Cartan-Hadamard manifold*, i.e. complete, connected, simply connected Riemannian manifold, with all sectional curvatures $K \leq 0$.
$\partial_\infty M$: *asymptotic boundary* (sphere at infinity), the set of all equivalence classes of unit speed geodesic rays:

$$\gamma_1 \sim \gamma_2 \iff sup_{t \geq 0} d\big(\gamma_1(t), \gamma_2(t)\big) < \infty.$$

Equivalently, $\partial_\infty M$ is the set of all unit speed geodesic rays starting from a fixed $o \in M$, hence we may interprete $\partial_\infty M = \mathbb{S}^{n-1} \subset T_o M$.
Compactification: $\overline{M} = M \cup \partial_\infty M$ equipped with the *cone topology*.
$\overline{M}$ homeomorphic to $\bar{B}^n \subset \mathbb{R}^n$, $\partial_\infty M$ homeomorphic to $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

Asymptotic Dirichlet problem

Given $\theta \in C(\partial_\infty M)$, find a (unique) $u \in C(\overline{M}) \cap C^\infty(M)$ such that

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{in } M, \\ u|\partial_\infty M = \theta. \end{cases}$$

Remark

In fact, we are looking for the *minimal submanifold* $S_u \subset M \times \mathbb{R}$, the graph of u, with prescribed "asymptotic boundary" $\{(y, \theta(y)) : y \in \partial_\infty M\} \subset \partial_\infty M \times \mathbb{R}$.

Remark

Of course, this is not always possible (ex. $\mathbb{R}^n$).

Curvature bounds

We consider this problem under curvature bounds

$$-b\big(r(x)\big)^2 \le K(P_x) \le -a\big(r(x)\big)^2 \quad (<0),$$

where $a, b\colon [0,\infty) \to (0,\infty)$ are smooth,
$r(x) = d(x,o),\ o \in M$ fixed,
and $K(P_x)$ is the sectional curvature of (a 2-plane) $P_x \subset T_xM$.
We want to find "optimal" curvature bounds a and b, in particular, we are interested in the curvature upper bound function a.

In Theorems below, the curvature bounds are assumed outside a compact set, i.e. $r(x) \geq R_0$ and "YES" means that the asymptotic Dirichlet problem for

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \quad \text{in } M, \\ u|\partial_\infty M = \theta. \end{cases}$$

is solvable with any continuous $\theta \in C(\partial_\infty M)$.

Earlier results

Theorem (Ripoll, Telichevesky (2013); Casteras, H., Ripoll (2013))

Suppose that

$$-r(x)^{-2-\varepsilon}\exp\big(2kr(x)\big) \le K(P_x) \le -k^2$$

for some constants $k,\ \varepsilon > 0$. *Then YES.*

Theorem (Casteras, H., Ripoll (2013))

Suppose that

$$-r(x)^{2(\phi-2)-\varepsilon} \le K(P_x) \le -\frac{\phi(\phi-1)}{r(x)^2}$$

for some constants $\phi > 1,\ 0 < \varepsilon < 2(\phi-1)$. *Then YES.*

Recent result

Theorem (Casteras, H., Ripoll (2015))

Suppose that $n = \dim M \geq 3$ *and*

$$-\frac{\left(\log r(x)\right)^{2\bar{\varepsilon}}}{r(x)^2} \leq K(P_x) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}$$

for some constants $\varepsilon > \bar{\varepsilon} > 0$. *Then YES.*

Remark

Curvature upper bound close to optimal.

Remark

These *same* curvature bounds apply to a large class of PDEs (Laplacian, p-Laplacian, ...) and are the best known bounds under which asymptotic Dirichlet problems are solvable.

Most recent result

Theorem (Casteras, Heinonen, H. (2015))

Suppose that

$$K(P_x) \leq -\frac{\phi(\phi-1)}{r(x)^2}, \quad \phi > 1,$$
$$|K(P_x)| \leq C|K(P'_x)|$$

for all 2-planes $P_x, P'_x \subset T_x M$ and that

$$\dim M = n > \frac{4}{\phi} + 1.$$

Then YES.

Comment on sharpness

Remark

A curvature upper bound alone is not sufficient for solvability.

Theorem (Ripoll, H. (2013))

There exists a 3-dimensional Cartan-Hadamard manifold M, with $K \leq -1$, such that the asymptotic Dirichlet problem is not solvable for any continuous (nonconstant) boundary data.

Sketch of proof

Let's recall:

Theorem (Casteras, H., Ripoll (2015))

Suppose that $n = \dim M \geq 3$ *and*

$$-\frac{\big(\log r(x)\big)^{2\bar{\varepsilon}}}{r(x)^2} \leq K(P_x) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)} =: -a\big(r(x)\big)^2$$

for some constants $\varepsilon > \bar{\varepsilon} > 0$*. Then YES.*

Given $\theta \in C(\partial_\infty M)$, interpret it as $\theta \in C(\mathbb{S}^{n-1})$, $\mathbb{S}^{n-1} \subset T_o M$ unit sphere.

Suppose θ is L-Lipschitz,

extend it radially to $\theta \in C(M \setminus \{o\})$. (In fact, we also smooth out θ.)

Then we have

$$|\nabla\theta(x)| \leq \frac{c}{f_a\big(r(x)\big)} \leq \frac{c}{r(x)\big(\log r(x)\big)^{1+\tilde{\varepsilon}}}, \quad 0 < \tilde{\varepsilon} < \varepsilon,$$

Sketch of proof

where f_a is the solution to Jacobi equation

$$\begin{cases} f_a'' &= a^2 f_a \\ f_a'(0) &= 1 \\ f_a(0) &= 0, \end{cases}$$

$a =$ the function in curvature upper bound.
Take a sequence $B_i = B(o, r_i),\ r_i \nearrow \infty$. Solve

$$\begin{cases} \operatorname{div} \dfrac{\nabla u_i}{\sqrt{1 + |\nabla u_i|^2}} = 0 \quad \text{in } B_i \\ u_i | \partial B_i = \theta. \end{cases}$$

Apply interior gradient estimates and regularity theory of elliptic PDEs to extract a converging subsequence

$$u_{i_k} \to u \quad \text{in } C^2_{\text{loc}}(M).$$

Sketch of proof

The limit u is a smooth solution to the minimal graph equation in M, so we are left with *the problem* to show

$$\lim_{x \to x_0} u(x) = \theta(x_0) \quad \forall x_0 \in \partial_\infty M.$$

Denote

$$h = \frac{|u - \theta|}{\nu}, \quad \nu \text{ sufficient large constant.}$$

Want to show

$$\varphi\big(h(x)\big) \to 0 \quad \text{as } x \to x_0 \in \partial_\infty M, \tag{1}$$

where φ is a smooth homeomorphism $\varphi\colon [0, \infty) \to [0, \infty)$ satisfying

$$\lim_{t \to 0} \frac{\varphi''(t)\varphi(t)}{\varphi'(t)^2} = 1.$$

Sketch of proof

Writing $\psi(t) = \varphi'(t)\varphi(t)$, we have

$$\varphi'(t)^2 \approx \frac{1}{2}\psi'(t)$$
$$\frac{\psi(t)^2}{\psi'(t)} \approx \frac{1}{2}\varphi(t)^2$$

for $t \approx 0$.
We show (1) (= $\varphi(h(x)) \to 0$) by proving:

$$\int_M \varphi(h)^2 < \infty$$

and

$$\sup_{B(x,r_0)} \varphi(h)^{2(n+1)} \leq c \int_{B(x,2r_0)} \varphi(h)^2.$$

Strategy of the proof

$$\left.\begin{array}{c}\text{Caccioppoli inequality}\\[2em]\text{Weighted Poincaré inequality}\end{array}\right\} \Longrightarrow$$

$$\int_M \varphi(h)^2 < \infty$$

$$\left.\begin{array}{c}\text{Caccioppoli inequality}\\[2em]\text{Sobolev inequality}\end{array}\right\} \overset{\text{Moser iteration}}{\Longrightarrow}$$

$$\sup_{B(x,r_0)} \varphi(h)^{2(n+1)} \le c \int_{B(x,2r_0)} \varphi(h)^2.$$

Caccioppoli inequality

Lemma

Suppose that $\Psi\colon [0,\infty) \to [0,\infty)$ *is a smooth homeomorphism,* $B = B(o,R)$, $\omega \geq 0$ *locally Lipschitz,* $\theta, u \in C(B) \cap W^{1,2}(B)$ *bounded,* $u \in C^2(B)$ *a solution to*

$$\operatorname{div} \frac{\nabla u}{W} = 0, \quad W = \sqrt{1 + |\nabla u|^2},$$

in B, $h = |u - \theta|/\nu$. *Suppose that* $\omega^2 \Psi(h) W \in W_0^{1,2}(B)$. *Then*

$$\int_B \omega^2 \Psi'(h) |\nabla u|^2 \leq 4 \int_B \omega^2 \Psi'(h) |\nabla \theta|^2 + 8\nu^2 \int_B \frac{\Psi^2}{\Psi'}(h) |\nabla \omega|^2 + 4\nu^2 \int_B \omega^2 \frac{\Psi^2}{\Psi'}(h) |\nabla \log W|^2.$$

Caccioppoli inequality

Proof.

Use

$$\eta = \nu\omega^2 W \left(\Psi\left(\tfrac{(u-\theta)^+}{\nu}\right) - \Psi\left(\tfrac{(u-\theta)^-}{\nu}\right) \right)$$

as a test function in

$$\int_B \frac{\langle \nabla u, \nabla \eta \rangle dx}{\sqrt{1 + |\nabla u|^2}} = 0.$$

□

Weighted Poincaré inequality

$$\left.\begin{array}{c} K \leq 0, \quad \text{everywhere} \\ \\ K(P_x) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \; r(x) \geq R_0 \end{array}\right\} \quad \overset{\text{Laplace comparison}}{\Longrightarrow}$$

$$r(x)\Delta r(x) \geq \begin{cases} n-1, & \text{everywhere} \\ \\ (n-1)\left(1 + \frac{1+\tilde{\varepsilon}}{\log r(x)}\right), & r(x) \geq R_1, \end{cases}$$

$0 < \tilde{\varepsilon} < \varepsilon, \; R_1 = R_1(\tilde{\varepsilon}) > R_0$.
Let $B = B(o, R), \; R \gg R_1, \; \theta \in C^\infty(M)$, and $u \in C^2(\bar{B})$ be the unique solution to the minimal graph equation with $u|\partial B = \theta|\partial B$.

Weighted Poincaré inequality

The estimates for $\Delta r(x)$, integration by parts, Hölder's inequality, etc. $\Longrightarrow$

$$n\Big(\int \varphi(h)^2 \underbrace{\big(\log(1+r)+\mathcal{C}(r)\big)}_{=:L(r)}\Big)^{1/2} \le 2\Big(\int \varphi'(h)^2 |\nabla h|^2 \omega^2\Big)^{1/2},$$

where $\mathcal{C}\colon [0,\infty) \to [0,\infty)$ is a bounded, smooth function, and

$$\omega = \frac{r\log(1+r)}{\sqrt{L(r)}}.$$

The idea is to estimate (modify) the RHS and absorb terms to the left:

$$\Big(\int \varphi'(h)^2 |\nabla h|^2 \omega^2\Big)^{1/2} \le c\Big(\int \underbrace{\psi'(h)}_{\sim\varphi'(h)^2} |\nabla u|^2 \omega^2\Big)^{1/2} + \frac{1}{\nu}\Big(\int \varphi'(h)^2 |\nabla \theta|^2 \omega^2\Big)^{1/2}$$

Sketch of proof

$$
\begin{aligned}
\Big(\int \varphi'(h)^2|\nabla h|^2\omega^2\Big)^{1/2} &\le c\Big(\int \underbrace{\psi'(h)}_{\sim\varphi'(h)^2}|\nabla u|^2\omega^2\Big)^{1/2} \\
&\quad + \frac{1}{\nu}\Big(\int \varphi'(h)^2|\nabla\theta|^2\omega^2\Big)^{1/2} \\
&\overset{\text{Cacc.}}{\le} c\Big(\int \varphi'(h)^2|\nabla\theta|^2\omega^2\Big)^{1/2} \\
&\quad + c\Big(\int \underbrace{\frac{\psi^2}{\psi'}(h)}_{\sim\varphi(h)^2}\underbrace{|\nabla\omega|^2}_{\le L(r)}\Big)^{1/2} \\
&\quad + c\Big(\int \varphi(h)^2\underbrace{|\nabla \log W|^2}_{=o(1/r^2)}\omega^2\Big)^{1/2}.
\end{aligned}
$$

Sketch of proof

Absorbing terms to the left, we get

$$\left(n - \sqrt{8}(1+\delta)\right)\Big(\int \varphi(h)^2 L(r)\Big)^{1/2} \le C + c\Big(\int \varphi'(h)^2 |\nabla\theta|^2 \omega^2\Big)^{1/2}, \quad (2)$$

$\delta > 0$ (as small as we wish by choosing ν large).
Then we use Young's inequality

$$ab = ka(b/k) \le kG(\sqrt{a})^2 + kF\big(\sqrt{b/k}\big),\ k > 0,$$

with complementary Young functions $F(\sqrt{\cdot})$ and $G(\sqrt{\cdot})^2$ satisfying

$$G \circ \varphi' = \varphi$$

and

$$F(t) \le \exp\left(-\frac{1}{t}\left(\log\frac{1}{t}\right)^{-1-\varepsilon_0}\right), \quad t > 0 \text{ small},$$

Sketch of proof

to further estimate the RHS:

$$\int \varphi'(h)^2|\nabla\theta|^2\omega^2 = \int \varphi'(h)^2|\nabla\theta|^2 L(r)\left(\frac{|\nabla\theta| r\log(1+r)}{L(r)}\right)^2$$
$$\leq k\int \underbrace{G\big(\sqrt{\varphi'(h)^2}\big)^2}_{=\varphi(h)^2} L(r)$$
$$+ k\int F\left(\frac{|\nabla\theta| r\log(1+r)}{\sqrt{k}L(r)}\right) L(r),\ k > 0 \text{ small.}$$

The first term on the right can be absorbed to the LHS of (2).

Sketch of proof

Finally we obtain

$$\int \varphi(h)^2 L(r) \leq C + c \int F\left(\frac{|\nabla\theta| r}{c}\right) L(r).$$

Combining estimates:

$$F(t) \leq \exp\left(-\frac{1}{t}\left(\log\frac{1}{t}\right)^{-1-\varepsilon_0}\right), \quad t > 0 \text{ small},$$

$$|\nabla\theta| \leq \frac{c}{r(\log r)^{1+\tilde{\varepsilon}}} \quad \text{from the curvature upper bound},$$

$$dV \leq f_b^{n-1} dr \wedge d\vartheta \quad \text{from curvature the lower bound},$$

we get

$$\int_{B(o,R)} \varphi(h)^2 L(r) \leq C < \infty,$$

with C independent of the radius R.

The role of curvature lower bound

Remark

The curvature lower bound was used in two places:

- To obtain the estimate

$$|\nabla \log W| = o(1/r), \quad r \to \infty, \quad W = \sqrt{1 + |\nabla u|^2}.$$

- To estimate the volume form

$$dV \leq f_b^{n-1} dr \wedge d\vartheta,$$

  where f_b is the solution to the Jacobi equation

$$\begin{cases} f_b'' &= b^2 f_b \\ f_b'(0) &= 1 \\ f_b(0) &= 0. \end{cases}$$

Most recent result

Let's recall

Theorem (Casteras, Heinonen, H. (2015))

Suppose that

$$K(P_x) \leq -\frac{\phi(\phi-1)}{r(x)^2}, \quad \phi > 1,$$
$$|K(P_x)| \leq C|K(P'_x)|$$

for all 2-planes $P_x, P'_x \subset T_xM$ and that

$$\dim M = n > \frac{4}{\phi} + 1.$$

Then YES.

Sketch of proof

Remark

Since there is no curvature lower bound, we do not have an estimate for $|\nabla \log W|$. Therefore, we must use another form of a Caccioppoli inequality.

Another Caccioppoli inequality

Lemma

Suppose that $\Psi\colon [0,\infty) \to [0,\infty)$ *is a smooth homeomorphism,* $B = B(o,R)$, $\omega \geq 0$ *locally Lipschitz,* $\theta, u \in C(B) \cap W^{1,2}(B)$ *bounded,* $u \in C^2(B)$ *a solution to*

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0$$

in B, $h = |u-\theta|/\nu$. *Suppose that* $\omega^2\Psi(h) \in W_0^{1,2}(B)$. *Then* $\forall \varepsilon > 0$

$$\int_B \omega^2 \Psi'(h) \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \leq C_\varepsilon \int_B \omega^2 \Psi'(h) |\nabla \theta|^2 + (4+\varepsilon)\nu^2 \int_B \frac{\Psi^2}{\Psi'}(h) |\nabla \omega|^2.$$

Another Caccioppoli inequality

We split the LHS into two parts and estimate:

$$\int_B \omega^2 \Psi'(h) \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \geq c_1 \int_{U_1} \omega^2 \Psi'(h) |\nabla u|^2 + c_2 \int_{U_2} \omega^2 \Psi'(h) |\nabla u|, \tag{3}$$

with

$$U_1 = \{|\nabla u| \leq \sigma\},\ U_1 = \{|\nabla u| > \sigma\},\ \sigma > 0 \text{ const.},$$

$$c_1 = \frac{1}{\sqrt{1+\sigma^2}}, \quad c_2 = \frac{1}{\sqrt{1+\sigma^{-2}}}.$$

Weighted Poincaré inequality

$$\left.\begin{array}{c} K \le 0, \quad \text{everywhere} \\ \\ K(P_x) \le -\frac{\phi(\phi-1)}{r(x)^2}, \; r(x) \ge R_0 \end{array}\right\} \overset{\text{Laplace comparison}}{\Longrightarrow}$$

$$r\Delta r(x) \ge \begin{cases} n-1, & \text{everywhere} \\ \\ \frac{(n-1)\phi}{1+\varepsilon} =: C_0, & r(x) \ge R_1, \end{cases}$$

$\varepsilon > 0, \; R_1 = R_1(\varepsilon) > R_0.$

Weighted Poincaré inequality

This time we obtain

$$(1 + C_0) \int_B \varphi(h) \leq C + c \int_B r\varphi'(h)|\nabla h|.$$

Estimating the RHS is more complicated. For instance, using (3), splitting $U_1 = U_3 \cup U_4$,

$$U_3 = \left\{|\nabla u| \leq \tilde{\sigma} \tfrac{\varphi(h)}{\varphi'(h)r}\right\}, \quad U_3 = \left\{\tilde{\sigma} \tfrac{\varphi(h)}{\varphi'(h)r} < |\nabla u| \leq \sigma\right\},$$

and using Caccioppoli (twice) we get

$$c \int_B \varphi(h) \leq C + c \int_B r\varphi'(h)|\nabla \theta| + c \int_B r^2 \varphi''(h)|\nabla \theta|^2.$$

To handle the last term on the right, we need another pair of complementary Young functions G_1 and F_1 such that

Sketch of proof

$G_1 \circ \varphi'' = \varphi$ and

$$F_1(t) \le ct \exp\left(-\frac{2^\lambda}{\sqrt{t}}\left(\log\frac{1}{t}\right)^{-\lambda}\right), \ \lambda > 1.$$

Putting all these together, we get

$$(C_0 - 4 - \varepsilon')\int_B \varphi(h) \le c + c\int_B F\big(r|\nabla\theta|\big) + c\int_B F_1\big(r^2|\nabla\theta|^2\big).$$

Curvature pinching condition $\Longrightarrow$

$$J(x) \le j(x)^C, \quad \text{where}$$

$$J(x) = \max\big|V\big(r(x)\big)\big|, \quad j(x) = \min\big|V\big(r(x)\big)\big|,$$

V Jacobi field along the geodesic ray γ from o to x,
$V_0 = 0,\ |V_0'| = 1,\ V \perp \dot\gamma$.

Sketch of proof

In particular,

$$|\nabla\theta(x)| \le \frac{c}{j(x)},$$
$$dV \le j(x)^{C(n-1)} dr \wedge d\vartheta$$

These together with estimates of the F and F_1 imply that

$$(C_0 - 4 - \varepsilon') \int_B \varphi(h) \le c + c\int_B F\big(r|\nabla\theta|\big) + c\int_B F_1\big(r^2|\nabla\theta|^2\big)$$
$$\le C < \infty,$$

with C independent of (the radius of) $B = B(o, R)$.

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Gràcies