M^n is a hypersurface and N is a unit normal vector field in a neighborhood of x, then

$$\nabla_{(\cdot)} N : T_x\Sigma \to T_x\Sigma \tag{1.30}$$

is a symmetric map (often referred to as the Weingarten map) and its eigenvalues $(\kappa_i)_{i=1,\dots,n-1}$ are called the principal curvatures. Moreover,

$$g(H,N) = -\sum_{i=1}^{n-1} \kappa_i \, . \tag{1.31}$$

Finally, if X is a vector field defined in a neighborhood of Σ, then the *divergence* of X at $x \in \Sigma$ is

$$\operatorname{div}_\Sigma X = \sum_{i=1}^{n-1} g(\nabla_{E_i} X, E_i)\, , \tag{1.32}$$

where E_i is an orthonormal basis for $T_x\Sigma$. Notice that $\operatorname{div}_\Sigma$ satisfies the Leibniz rule

$$\operatorname{div}_\Sigma(fX) = \langle \nabla_\Sigma f, X\rangle + f \operatorname{div}_\Sigma(X)\, . \tag{1.33}$$

We can also use $\operatorname{div}_\Sigma$ to define the Laplace operator Δ_Σ on Σ by

$$\Delta_\Sigma f = \operatorname{div}_\Sigma(\nabla_\Sigma f)\, . \tag{1.34}$$

A function f is said to be *harmonic* on Σ if $\Delta_\Sigma f = 0$.

Remark 1.3. *Note that*

$$\begin{aligned} \operatorname{div}_\Sigma Y^N &= \sum_i g(E_i, \nabla_{E_i} Y^N) = -\sum_i g(Y^N, \nabla_{E_i} E_i) \\ &= -g(Y^N, H)\, . \end{aligned} \tag{1.35}$$

1.3. The first variation formula. Let $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be a variation of Σ with compact support and fixed boundary. That is, $F = \text{Id}$ outside a compact set,

$$F(x,0) = x\, , \tag{1.36}$$

and for all $x \in \partial\Sigma$,

$$F(x,t) = x\, . \tag{1.37}$$

The vector field F_t restricted to Σ is often called the *variational vector field.* Now we want to compute the first variation of area for this one-parameter family of surfaces. Let x_i be local coordinates on Σ. Set

$$g_{ij}(t) = g(F_{x_i}, F_{x_j})\, , \tag{1.38}$$

$$\nu(t) = \sqrt{\det(g_{ij}(t))}\,\sqrt{\det(g^{ij}(0))}\, , \tag{1.39}$$

where a^{ij} denotes the inverse of the matrix a_{ij}. Note that $\nu(t)$ is well defined, independent of the choice of a coordinate system on Σ (since $\det(g_{ij}(t))$ changes by the determinant squared of the differential of a coordinate transformation while $\det(g^{ij}(0))$ changes by the inverse of this). Furthermore, the area formula is

$$\operatorname{Vol}(F(\Sigma,t)) = \int \nu(t)\,\sqrt{\det(g_{ij}(0))}\, , \tag{1.40}$$

where the integral is over Σ. Differentiating this gives

$$\frac{d}{dt}_{t=0} \operatorname{Vol}(F(\Sigma,t)) = \int \frac{d}{dt}_{t=0} \nu(t)\,\sqrt{\det(g_{ij}(0))}\, . \tag{1.41}$$

To evaluate $d/dt_{t=0}\nu(t)$ at some point x, we may choose the coordinate system such that at x it is orthonormal, i.e., so that at the point x,

$$g_{ij}(0) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j\, , \\ 1 & \text{if } i = j\, . \end{cases} \tag{1.42}$$

Using this and the fact that the t and x_i derivatives commute (i.e., $\nabla_{F_t} F_{x_i} - \nabla_{F_{x_i}} F_t = [F_t, F_{x_i}] = 0$), we get at x,

$$\begin{aligned} \frac{d}{dt}_{t=0} \nu(t) &= \frac{1}{2}\sum_{i=1}^k \frac{d}{dt}\langle F_{x_i}, F_{x_i}\rangle = \sum_{i=1}^k \langle \nabla_{F_t} F_{x_i}, F_{x_i}\rangle \\ &= \sum_{i=1}^k \langle \nabla_{F_{x_i}} F_t, F_{x_i}\rangle = \operatorname{div}_\Sigma F_t\, . \end{aligned} \tag{1.43}$$

We can relate this formula to the mean curvature by writing the vector field F_t as the sum of its normal and tangential parts to get

$$\begin{aligned} \frac{d}{dt}_{t=0} \nu(t) &= \sum_{\ell=1}^{n-k}\sum_{i=1}^{k} \langle \nabla_{F_{x_i}} \langle F_t, N_\ell\rangle\, N_\ell, F_{x_i}\rangle + \operatorname{div}_\Sigma F_t^T \\ &= \sum_{\ell=1}^{n-k}\sum_{i=1}^{k} \langle F_t, N_\ell\rangle\, \langle \nabla_{F_{x_i}} N_\ell, F_{x_i}\rangle + \operatorname{div}_\Sigma F_t^T \\ &= -\langle F_t, H\rangle + \operatorname{div}_\Sigma F_t^T\, . \end{aligned} \tag{1.44}$$

Here N_ℓ is an orthonormal basis for the normal bundle of Σ at x. Integrating (1.43) and (1.44) gives the so-called first variation formula:

$$\frac{d}{dt}_{t=0} \operatorname{Vol}(F(\Sigma,t)) = -\int_\Sigma \langle F_t, H\rangle = \int_\Sigma \operatorname{div}_\Sigma F_t\, . \tag{1.45}$$

Note that Stokes' theorem was used to see that $\int \operatorname{div}_\Sigma F_t^T = 0$. As a consequence of (1.45), we see that Σ is a critical point for the area functional if and only if the mean curvature H vanishes identically.