Minimal Surfaces

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Contents

The material is collected mainly from books [CM], [EG], [GT], [GM], [G], [O] and from survey articles [MP1] and [MP2].

1 Introduction

We start by recalling some background, history, origin, etc. of the theory of minimal surfaces. Some notions that appear in the introduction will be defined and studied later in detail.

1.1 Background, history, origin, etc.

The (mathematical) theory of minimal surfaces in $\mathbb{R}^3$ has its origin in calculus of variations developed by Euler and Lagrange in the 18th century.

Lagrange (Joseph-Louis Lagrange, Giuseppe Lodovico (Luigi) Lagrangia, 1736-1813) studied the variational problem of finding a surface

$$S = \{(x, y, u(x, y))\}$$

of least area bounded by a closed curve and derived the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla|^2}}\right) = 0,$$

the so-called *Euler-Lagrange* equation, for the solution u. We will call this equation the *minimal graph equation*. However, he did not succeed in finding any solutions (other than the plane). This problem of finding the surface of least area is nowadays known as the *Plateau problem* named after the (blind) Belgian physicist Joseph Plateau (1801-1883) who made experiments with soap films and bubbles.

In 1776 Jean Baptiste Meusnier (1754-1793) discovered that *helicoid* and *catenoid* satisfy the minimal graph equation (locally) and that surfaces with zero *mean curvature* are area-minimizing. (Catenoid was discovered by Euler in 1744.)

Scherk (Heinrich Scherk, 1798-1885) constructed two complete embedded minimal surfaces (doubly periodic and singly periodic). These were the third non-trivial examples of minimal surfaces (after helicoid and catenoid). Scherk's doubly periodic surface is defined over "the white squares of the chessboard" with vertical lines at corners. A fundamental "piece" of Scherk's surface is the graph of the function

$$u(x, y) = \log \frac{\cos y}{\cos x}$$

over the square $\{(x, y)\colon |x| < \pi/2, |y| < \pi/2\}$.

Schwarz (Hermann Schwarz, 1843-1921) solved the Plateau problem for quadrilaterals and together with his student E.R. Neovius (Edvard Rudolf Neovius (uncle of Rolf Nevanlinna)) described periodic minimal surfaces.

Weierstrass (Karl Theodor Wilhem Weierstrass, 1815-1897) and Enneper (Alfred Enneper, 1830-1885) developed representation formulas which give link to complex analysis. (See Enneper's surface).

It is also worth mentioning that Lebesgue (Henri Léon Lebesgue, 1875-1941) developed the theory of measure and integral and studied the Plateau problem in his thesis (1902).

Complete solution to the Plateau problem (in 3-space) was obtained in 1931 and 1930 by Jesse Douglas (1897-1965, Fields medal in 1936) and Tibor Radó (1895-1965).

Bernstein-type problems deal with codimension 1 minimal (hyper)surfaces in $\mathbb{R}^n$ that are graphs of entire functions $u\colon \mathbb{R}^{n-1} \to \mathbb{R}$ solving the minimal graph equation. The question here was whether the function u must be affine (and hence the surface is a codimension 1 hyperplane). This is the case in dimensions $n \leq 8$ but false in dimensions $n \geq 9$. The problem is named after Sergei Natanovich Bernstein (1880-1968) who proved in 1915-1917 that a function $u\colon \mathbb{R}^2 \to \mathbb{R}$ must be affine if it is a (global) solution to the minimal graph equation. In 1962 Fleming gave another proof by showing that there are no non-planar area-minimizing cones in $\mathbb{R}^3$. De Giorgi proved in 1965 that if there are no non-planar area-minimizing cones in $\mathbb{R}^{n-1}$, then the analogue of Bernstein's theorem is true in $\mathbb{R}^n$, in particular, Bernstein's theorem is true in $\mathbb{R}^4$. Almgren (1966) extended Bernstein's theorem to $\mathbb{R}^5$ by showing that there are no non-planar minimizing cones in $\mathbb{R}^4$. Simons (1968) extended Almgren's result up to dimension 7, thus extending Bernstein's theorem to $\mathbb{R}^n, n \leq 8$. He also gave examples of locally stable cones in $\mathbb{R}^8$ and asked if they were globally area-minimizing. Finally, Bombieri, De Giorgi, and Giusti (1969) showed that Simons' cones are indeed globally minimizing, and showed that in $\mathbb{R}^n$, $n \geq 9$, there are graphs that are minimal but not hyperplanes.

In 1982 Celso Costa disproved the conjecture that the plane, catenoid, and helicoid are the only complete, embedded minimal surfaces in $\mathbb{R}^3$ of finite topological type (i.e. homeomorphic with the interior of compact surface with boundary). He constructed a (complete, embedded) minimal surface which is topologically a thrice punctured torus and has two catenoidal ends, one planar end, and has total curvature -12π.

1.2 Some examples

Here we recall the examples of minimal surfaces already mentioned in the previous subsection.

(i) The plane, $z = u(x, y) \equiv 0$.

(ii) The helicoid, $z = u(x, y) = \arctan(y/x) = \tan^{-1}(y/x)$. In parametric form it is given by

$$(t, s) \mapsto (t\cos s, t\sin s, s),\ \ s, t \in \mathbb{R}.$$

It is a complete, embedded, singly-periodic, simply connected ruled surface, with infinite total curvature. Catalan (1842) showed that it is the only non-flat ruled minimal surface.

(iii) The Catenoid, $z = \cosh^{-1}\sqrt{x^2 + y^2}$. Thus it is obtained by rotating the curve $x = \cosh z$ around the z-axis. The catenoid is the only non-flat minimal surface of revolution. It is complete, embedded, of finite total curvature, and topologically an annulus (genus zero, two ends).

(iv) Scherk's (doubly periodic) surface is the union of the closures of surfaces

$$\Sigma_{k,\ell} = \left\{(x, y, z)\colon |x - k| < 1, |y - \ell| < 1, z = \log\frac{\cos\frac{\pi}{2}(y-\ell)}{\cos\frac{\pi}{2}(x-k)}\right\},$$

where $k, \ell \in \mathbb{Z}$, with $k + \ell \equiv 0 \mod 0$.

(v) Enneper's surface is parameterized by

$$(s, t) \mapsto (s - s^3/3 + st^2, -t - s^2t + t^3/3, s^2 - t^2),\ \ s, t \in \mathbb{R}.$$

It is a non-embedded minimal surface with finite total curvature.

1.3 Weierstrass-Enneper parameterization

The Weierstrass-Enneper parameterization is an effective way to produce (parameterized) minimal surfaces. It also gives a link between minimal surfaces and complex analysis. Let f and g be functions defined either on the entire complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$, where g is meromorphic and f is analytic such that fg^2 is analytic (if g has a pole of order m, then f has a zero of order $2m$). Then the surface parameterized by

$$\zeta \mapsto \big(x_1(\zeta), x_2(\zeta), x_3(\zeta)\big)$$

is minimal if

$$x_k(\zeta) = \Re\left(\int_0^\zeta \varphi_k(z)dz\right) + c_k, \;\; k = 1, 2, 3,$$

where $\Re(z) = u$ denotes the real part of a complex number $z = u + iv$, $c_k \in \mathbb{R}$ is constant, and

$$\begin{aligned}\varphi_1 &= f(1 - g^2)/2,\\ \varphi_2 &= if(1 + g^2)/2,\\ \varphi_3 &= fg\end{aligned}$$

are analytic functions. We will prove this later. Also the converse is true: every simply connected minimal surface in $\mathbb{R}^3$ has a parameterization of this type. For instance, Enneper's surface is obtained by choosing $f(z) = 1$ and $g(z) = z$.

1.4 Equivalent definitions

Here we present several equivalent definitions for minimal surfaces. The variety of these definitions shows that minimal surfaces are related to many different fields in mathematics. We will prove some of these equivalences later.

1. *Local area-minimizing definition.* A surface $M \subset \mathbb{R}^3$ is minimal if and only if every point $p \in M$ has a neighborhood with least area relative to its boundary.

2. *Variational definition.* A surface $M \subset \mathbb{R}^3$ is minimal if and only if it is a critical point of the area functional for all compactly supported variations.

3. *Soap film definition.* A surface $M \subset \mathbb{R}^3$ is minimal if and only if every point $p \in M$ has a neighborhood U_p which is equal to the idealized soap film with boundary ∂U_p.

4. *Mean curvature definition.* A surface $M \subset \mathbb{R}^3$ is minimal if and only if its mean curvature vanishes identically.

5. *PDE definition.* A surface $M \subset \mathbb{R}^3$ is minimal if and only if it can be expressed locally (and after a rotation) as a graph $(x, y, u(x, y))$ of a solution u to

$$(1 + u_x^2)u_{yy} - 2u_xu_yu_{xy} + (1 + u_y^2)u_{xx} = 0.$$

   This is just an equivalent way to say that u solves the minimal graph equation.

6. *Energy definition.* A conformal immersion $X \colon M \to \mathbb{R}^3$ is minimal if and only if it is a critical point of the Dirichlet energy for all compactly supported variations, or equivalently if every point $p \in M$ has a neighborhood with least energy relative to its boundary.

7. *Harmonic definition.* If $X = (x_1, x_2, x_3) \colon M \to \mathbb{R}^3$ is an isometric immersion of a Riemannian 2-manifold (or a Riemann surface) into $\mathbb{R}^3$, then X (and M) is said to be minimal if each coordinate function x_i is a harmonic function on M.

8. *Gauss map definition.* A surface $M \subset \mathbb{R}^3$ is minimal if and only if its stereographically projected Gauss map $M \to \mathbb{C} \cup \{\infty\}$ is meromorphic with respect to the underlying Riemann surface structure of M, and M is not a piece of a sphere.

9. *Mean curvature flow definition.* Minimal surfaces are the critical points of the mean curvature flow.

Remarks 1.5. The condition 1 above is a local property: there might be other surfaces with less area but the same boundary. Definitions 6 and 7 relate minimal surfaces to harmonic functions and potential theory. Furhermore, definition 7 and the maximum principle for harmonic functions imply that there are no compact, complete minimal surfaces in $\mathbb{R}^3$.

1.6 Minimal graph equation

Suppose that $u \colon \Omega \to \mathbb{R}$ is a C^2-function defined on an open (bounded) subset of the plane $\mathbb{R}^2$. Denote by $\Gamma_u \subset \mathbb{R}^3$ its graph

$$\Gamma_u = \{\big(x, y, u(x, y)\big) \colon (x, y) \in \Omega\}.$$

It is a 2-dimensional submanifold of $\mathbb{R}^3$ and the tangent space (plane) $T_p\Gamma_u$ at $p = (x, y, u(x, y)) \in \Gamma_u$ is spanned by vectors $(1, 0, u_x)$ and $(0, 1, u_y)$, where u_x and u_y denote the partial derivative of u with respect to x and y, respectively. The absolute value of the cross product $(1, 0, u_x) \times (0, 1, u_y)$ is the area of the parallelogram spanned by $(1, 0, u_x)$ and $(0, 1, u_y)$, and so the area of the graph is

$$\begin{aligned} \operatorname{Area}(\Gamma_u) &= \int_\Omega |(1, 0, u_x) \times (0, 1, u_y)| = \int_\Omega \sqrt{1 + u_x^2 + u_y^2} \\ &= \int_\Omega \sqrt{1 + |\nabla u|^2}. \end{aligned}$$

Let $\eta \in C_0^2(\Omega)$. Then the graphs of u and $u + t\eta$, $t \in \mathbb{R}$, have the same "boundary" $\partial\Gamma_u = \{(x, y, u(x, y) \colon (x, y) \in \partial\Omega\}$ and

$$\operatorname{Area}(\Gamma_{u+t\eta}) = \int_\Omega \sqrt{1 + |\nabla u + t\nabla\eta|^2}.$$

Differentiating with respect to t and using Green's formula we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Area}(\Gamma_{u+t\eta})_{|t=0} &= \frac{d}{dt} \int_\Omega \sqrt{1 + |\nabla u + t\nabla\eta|^2}_{|t=0} \\ &= \int_\Omega \frac{d}{dt} \sqrt{1 + |\nabla u + t\nabla\eta|^2}_{|t=0} \\ &= \int_\Omega \frac{1}{2}(1 + |\nabla u|^2)^{-1/2} \frac{d}{dt} \big\langle \nabla(u + t\eta), \nabla(u + t\eta) \big\rangle_{|t=0} \\ &= \int_\Omega \frac{\langle \nabla u, \nabla\eta \rangle}{\sqrt{1 + |\nabla u|^2}} \\ &= -\int_\Omega \eta \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \end{aligned}$$

We say that u is a *critical point for the area functional* if the derivative above at $t = 0$ is zero. In that case, since the last integral vanishes for all $\eta \in C_0^2(\Omega)$, we conclude that $u \in C^2(\Omega)$ is a critical point if and only if it satisfies the minimal graph equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{1.7}$$

Next we show that a critical point for the area functional, in fact, minimizes the area among surfaces in the cylinder $\Omega \times \mathbb{R}$ with the same boundary $\partial\Gamma_u$. For that and later purposes we note that the unit vector

$$N = \frac{(1,0,u_x)\times(0,1,u_y)}{|(1,0,u_x)\times(0,1,u_y)|} = \frac{(1,0,u_x)\times(0,1,u_y)}{\sqrt{1+|\nabla u|^2}}$$

is orthogonal to both $(1,0,u_x)$ and $(0,1,u_y)$, and therefore it is the (upwards pointing) normal to Γ_u. We define a 2-form ω in the cylinder $\Omega \times \mathbb{R}$ by setting

$$\omega(X,Y) = \det(X,Y,N)$$

for vectors $X, Y \in \mathbb{R}^3$. Note that ω is the contraction by N of the standard volume form $\tilde\omega = dx \wedge dy \wedge dz$, i.e. $\omega = N \lrcorner \tilde\omega = i_N\tilde\omega$. Hence ω is the volume (area) form of Γ_u. Since $\omega = a dx \wedge dy + b dx \wedge dz + c dy \wedge dz$ and

$$\begin{aligned} a &= \omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 1/\sqrt{1+|\nabla u|^2}, \\ b &= \omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = u_y/\sqrt{1+|\nabla u|^2}, \\ c &= \omega(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = -u_x/\sqrt{1+|\nabla u|^2}, \end{aligned}$$

we see that

$$\omega = \frac{dx \wedge dy - u_x dy \wedge dz - u_y dz \wedge dx}{\sqrt{1+|\nabla u|^2}}.$$

Furthermore, since u satisfies the minimal graph equation we obtain

$$d\omega = \left\{\frac{\partial}{\partial x}\left(\frac{-u_x}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial y}\left(\frac{-u_y}{\sqrt{1+|\nabla u|^2}}\right)\right\} dx \wedge dy \wedge dz = 0.$$

Thus ω is a closed 2-form in the cylinder $\Omega \times \mathbb{R}$. Let then Σ be another (smooth) surface (non necessarily a graph) in $\Omega \times \mathbb{R}$ with the same boundary than Γ_u ($\partial\Gamma_u = \partial\Sigma$.) Then Σ and Γ_u bound an open set $U \subset \mathbb{R}^3$ where $d\omega = 0$. The set U may have several components but applying Stokes' theorem in each component we obtain

$$\int_{\Gamma_u} \omega = \int_\Sigma \omega.$$

On the other hand, by definition $|\omega(X,Y)| = |\det(X,Y,N)|$ is the volume of the polyhedron spanned by vectors X, Y, and N. In particular, for any unit vectors X and Y,

$$|\omega(X,Y)| \le 1,$$

with the equality if and only if X, Y, and N are orthonormal. Hence

$$\text{Area}(\Gamma_u) = \int_{\Gamma_u} \omega = \int_{\Sigma} \omega \leq \text{Area}(\Sigma). \tag{1.8}$$

This shows that Γ_u minimizes the area among such surfaces (inside $\Omega \times \mathbb{R}$). If Ω is convex, then Γ_u is area-minimizing among all surfaces $\Sigma \subset \mathbb{R}^3$ with $\partial\Sigma = \partial\Gamma_u$. To see this, let Σ be such a surface and let $P\colon \mathbb{R}^3 \to \Omega \times \mathbb{R}$ be the nearest point projection. The convexity of Ω implies that P is 1-Lipschitz map that is equal to the identity on $\Omega \times \mathbb{R}$. In particular, $\text{Area}(P\Sigma) \leq \text{Area}(\Sigma)$. Applying (1.8) to $P\Sigma$ we obtain

$$\text{Area}(\Gamma_u) \leq \text{Area}(P\Sigma) \leq \text{Area}(\Sigma).$$

Suppose that B_r is an open ball in $\mathbb{R}^3$ whose projection to the plane is contained in Ω. Then $\partial B_r \cap \Gamma_u$ (if non-empty) divides the sphere ∂B_r into two components at least one of which has the area at most $\text{Area}(\mathbb{S}^2)r^2/2$, where $\mathbb{S}^2$ is the unit 2-sphere. By (1.8), we obtain an estimate

$$\text{Area}(B_r \cap \Gamma_u) \leq \text{Area}(\mathbb{S}^2)r^2/2. \tag{1.9}$$

The fact that there are at most two components follows also from the minimality.

1.10 Bernstein's example

We prove that the area functional need not have a minimizer. Consider the annulus

$$\Omega = \{x \in \mathbb{R}^2 \colon \rho < |x| < R\}$$

and fix the boundary values g,

$$g(x) = \begin{cases} m, & \text{if } |x| = \rho, \\ 0, & \text{if } |x| = R. \end{cases}$$

Suppose that u is a minimizer for

$$\int_{\Omega} \sqrt{1 + |\nabla u|^2}$$

with boundary values g. Then u is a solution to (1.7) and later we will prove that a solution (if exists) is unique. It follows that u is radial, i.e. $u(x) = u(|x|)$ (Exercise). Thus the area of the graph is

$$\text{Area}(\Gamma_u) = 2\pi \int_{\rho}^{R} r\sqrt{1 + u_r^2} dr.$$

The corresponding Euler-Lagrange equation is

$$u_{rr} = -\frac{1}{r}(u_r + u_r^3)$$

and u solves this equation. Hence

$$u(r) = c \log \frac{R + \sqrt{R^2 - c^2}}{r + \sqrt{r^2 - c^2}},$$

where $c \in [0, \rho]$ is a constant such that $u(\rho) = m$. On the other hand,

$$m = u(\rho) = c \log \frac{R + \sqrt{R^2 - c^2}}{\rho + \sqrt{\rho^2 - c^2}} \leq \rho \log \frac{R + \sqrt{R^2 - \rho^2}}{\rho + \sqrt{\rho}} =: m(R, \rho).$$

Hence the problem can have a solution, and thus a minimizer can exist, only if $m \leq m(R, \rho)$.

2 Geometry of submanifolds of $\mathbb{R}^{n+k}$

2.1 The standard connection of $\mathbb{R}^m$

We denote by

$$\partial_i = \frac{\partial}{\partial x_i},\ i = 1, \dots, m,$$

the standard basis of $\mathbb{R}^m$. Thus these vectors are orthonormal with respect to the usual inner product which we denote by $\langle \cdot, \cdot \rangle$. A vector field defined on an open set $\Omega \subset \mathbb{R}^m$ is a mapping $V \colon \Omega \to \mathbb{R}^m$ which we write as

$$V_p = V(p) = \sum_{i=1}^m v^i(p)\partial_i,$$

where $v^i \colon \Omega \to \mathbb{R},\ i = 1, \dots, m$, are (component) functions. Vector fields act on smooth functions f as

$$Vf = \sum_{i=1}^m v^i(p)\partial_i f, \quad \partial_i f = \frac{\partial f}{\partial x_i}.$$

Definition 2.2. Let X and V be vector fields such that V is smooth (i.e. the component functions v^i are smooth). Then the *covariant derivative* of V in the direction X_p is the vector

$$\left(\bar\nabla_X V\right)_p = (X_p v^1, X_p v^2, \dots, X_p v^m) \in \mathbb{R}^m$$

and $\bar\nabla_X V$ is the vector field $p \mapsto \left(\bar\nabla_X V\right)_p$.

We denote by $\mathcal{T}(\Omega)$ the set of all smooth vector fields on $\Omega \subset \mathbb{R}^m$.

Definition 2.3. The mapping

$$\bar\nabla \colon \mathcal{T}(\Omega) \times \mathcal{T}(\Omega) \to \mathcal{T}(\Omega), \quad \bar\nabla(X, Y) = \bar\nabla_X Y,$$

is called the *Levi-Civita connection* on Ω. We also call it the standard connection on $\Omega \subset \mathbb{R}^m$.

The standard connection has the following properties:

1. $\bar\nabla_X Y$ is C^∞-linear in X: for every functions $f, g \in C^\infty(\Omega)$ and vector fields $X, Y, V \in \mathcal{T}(\Omega)$

$$\bar\nabla_{fX+gY} V = f\bar\nabla_X V + g\bar\nabla_Y V;$$

2. $\bar\nabla_X Y$ is $\mathbb{R}$-linear in Y: for every $a, b \in \mathbb{R}, X, Y, V \in \mathcal{T}(\Omega)$

$$\bar\nabla_X(aY + bV) = a\bar\nabla_X Y + b\bar\nabla_X V;$$

3. $\bar\nabla$ satisfies the Leibniz rule: for every $f \in C^\infty(\Omega), X, Y \in \mathcal{T}(\Omega)$

$$\bar\nabla_X(fY) = f\bar\nabla_X Y + (Xf)Y;$$

4. $\bar\nabla$ is *torsion-free*: for every $X, Y \in \mathcal{T}(\Omega)$

$$\bar\nabla_X Y - \bar\nabla_Y X = [X, Y],$$

where $[X, Y] \in \mathcal{T}(\Omega)$ is the Lie bracket

$$[X, Y]f = X(Yf) - Y(Xf);$$

5. $\bar\nabla$ is *compatible* with the standard inner product $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^m$: for every $X, Y, Z \in \mathcal{T}(\Omega)$

$$X\langle Y, Z\rangle = \langle \bar\nabla_X Y, Z\rangle + \langle Y, \bar\nabla_X Z\rangle.$$

The standard connection $\bar\nabla$ is the unique mapping $\mathcal{T}(\Omega)\times\mathcal{T}(\Omega)\to\mathcal{T}(\Omega)$ satisfying the properties above.

2.4 The Riemannian structure on a submanifold of $\mathbb{R}^m$

Let $\Omega\subset\mathbb{R}^n$ be an open set and $\varphi\colon\Omega\to\mathbb{R}^m$ a smooth mapping. Recall that φ is an *immersion* if the differential $d\varphi(x)\colon\mathbb{R}^n\to\mathbb{R}^m$ is injective for all $x\in\Omega$. If φ is one-to-one, the image $M=\varphi\Omega\subset\mathbb{R}^m$ is called an *immersed submanifold* of $\mathbb{R}^m$. If, in addition, φ is a homeomorphism onto $\varphi\Omega\subset\mathbb{R}^m$, then φ is an *embedding* and $M=\varphi\Omega$ is an n-dimensional submanifold of $\mathbb{R}^m$. Note that here M has the relative topology. In general, a smooth manifold $M\subset\mathbb{R}^m$ is a *submanifold* of $\mathbb{R}^m$ if the inclusion $\pi\colon M\hookrightarrow\mathbb{R}^m$, $\pi(x)=x$, is an embedding. [We use the notation $\pi=(\pi_1,\pi_2,\ldots,\pi_m)$ for the inclusion because then $\pi_i\colon M\to\mathbb{R}$ will be the projection to the x_i-axis.]

If φ is an immersion, every point $x\in\Omega$ has a neighborhood $U\subset\Omega$ such that $\varphi|U$ is an embedding.

Let $M\subset\mathbb{R}^m$ be a smooth n-dimensional submanifold of $\mathbb{R}^m$. Thus locally M can be parametrized by a smooth homeomorphism $\varphi\colon\Omega\to U$, where $\Omega\subset\mathbb{R}^n$ and $U\subset M$ are open, and the differential $d\varphi(x)$ at x is of rank n for every $x\in\Omega$. We identify the tangent space $T_pM, p\in U$, with the image $d\varphi\big(\varphi^{-1}(p)\big)\mathbb{R}^n$. Thus T_pM is an n-dimensional vector subspace of $\mathbb{R}^m$. Each T_pM inherits an inner product $\langle\cdot,\cdot\rangle$ from $\mathbb{R}^m$: for every vectors $v,w\in T_pM$,

$$\langle v,w\rangle = v\cdot w,$$

where $v\cdot w$ is just the standard inner product in $\mathbb{R}^m$. This induced inner product $\langle\cdot,\cdot\rangle$ defines the Riemannian metric (and thus the Riemannian submanifold structure) on M. For every $p\in M$, the inner product of $\mathbb{R}^m$ splits $\mathbb{R}^m$ orthogonally into

$$T_pM\oplus T_pM^\perp.$$

We write $N_pM=T_pM^\perp$ and call it the *normal space* of M at p. Furthermore, we denote by

$$TM=\bigsqcup_{p\in M}T_pM \quad\text{and}\quad NM=\bigsqcup_{p\in M}N_pM$$

the tangent and normal bundles, respectively.

Next we want to define a (in fact, the) Levi-Civita connection ∇ on M that satisfies conditions 1.-5. above, in particular, that is compatible with the induced Riemannian metric. Let $\tilde X,\tilde Y\in\mathcal{T}(\Omega)$ be smooth vector fields in an open set $\Omega\subset\mathbb{R}^m$. Then at every $p\in\Omega$

$$\big(\bar\nabla_{\tilde X}\tilde Y\big)_p=\big(\bar\nabla_{\tilde X}\tilde Y\big)_p^\top+\big(\bar\nabla_{\tilde X}\tilde Y\big)_p^\perp,$$

where

$$\big(\bar\nabla_{\tilde X}\tilde Y\big)_p^\top\in T_pM \quad\text{and}\quad \big(\bar\nabla_{\tilde X}\tilde Y\big)_p^\perp\in N_pM.$$

Definition 2.5. The Levi-Civita connection ∇ of M is simply the orthogonal projection on TM of the standard connection of $\mathbb{R}^m$. More precisely, let $X,Y\in\mathcal{T}(M)$ be smooth vector fields, i.e. at each point $p\in M$

$$X_p=\sum_{i=1}^m a^i(p)\partial_i,\quad Y_p=\sum_{i=1}^m b^i(p)\partial_i,$$

where $a^i, b^i : M \to \mathbb{R}$ are smooth functions. For each $p \in M$, let $\tilde{X}$ and $\tilde{Y}$ be (any) smooth extensions of X and Y to a neighborhood (in $\mathbb{R}^m$) of p. Then we define

$$(\nabla_X Y)_p = \big(\bar{\nabla}_{\tilde{X}} \tilde{Y}\big)_p^\top \in T_pM,$$

where

$$\big(\bar{\nabla}_{\tilde{X}} \tilde{Y}\big)_p^\top$$

is the orthogonal projection of $\big(\bar{\nabla}_{\tilde{X}} \tilde{Y}\big)_p$ to T_pM.

The properties 1.-3. are clearly true for ∇ and we leave it as an exercise to verify that ∇ is torsion-free and compatible with the induced inner product (Riemannian metric). Note that $\nabla_X Y$ is well-defined, i.e. does not depend on the extensions $\tilde{X}$ and $\tilde{Y}$. This holds since

$$\big(\bar{\nabla}_{\tilde{X}} \tilde{Y}\big)_p$$

depends only on $X_p = \tilde{X}_p$ and values of $\tilde{Y}$ along any path $\gamma :]-\varepsilon, \varepsilon[\to \mathbb{R}^m$, with $\gamma_0 = p$ and $\dot{\gamma}_0 = X_p$. In particular, γ can be taken as a path $\gamma :]-\varepsilon, \varepsilon[\to M$ along M.

2.6 The gradient, divergence, and the Laplacian on M

Let $f : M \to \mathbb{R}$ be a C^1-function and $X \in T_pM$. Then

$$Xf = (f \circ \gamma)'(0),$$

where $\gamma :]-\varepsilon, \varepsilon[\to M$ is any C^1-path, with $\gamma(0) = p$ and $\dot{\gamma}_0 = X$. The *gradient* of f is defined as

$$\nabla^M f(p) = \sum_{i=1}^n (X_i f) X_i,$$

where $\{X_i\}_{i=1}^n$ is an orthonormal basis of T_pM. In particular, if f is a C^1-function in a neighborhood (in $\mathbb{R}^m$) of p, then

$$\nabla^M f(p) = \big(\nabla f(p)\big)^\top,$$

where

$$\nabla f(p) = \sum_{i=1}^m \partial_i f(p) \partial_i$$

is the standard gradient (in $\mathbb{R}^m$) of f. Given a chart $\varphi : U \to \mathbb{R}^n$, $U \subset M$, and the corresponding local parametrization $F = \varphi^{-1} : \varphi U \to U$ we can write $\nabla^M f$ in U as

$$\nabla^M f = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial F}{\partial x^j},$$

where $g^{ij} : U \to \mathbb{R}$, $\frac{\partial f}{\partial x^i} : U \to \mathbb{R}$, and $\frac{\partial F}{\partial x^j} : U \to TM$ are defined as

$$\begin{aligned}
\frac{\partial f}{\partial x^i}(p) &= \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}\big(\varphi(p)\big), \\
\frac{\partial F}{\partial x^j}(p) &= \left(\frac{\partial F_1}{\partial x^j}\big(\varphi(p)\big), \dots, \frac{\partial F_m}{\partial x^j}\big(\varphi(p)\big) \right) \in T_pM, \\
g_{ij}(p) &= \frac{\partial F}{\partial x^i}(p) \cdot \frac{\partial F}{\partial x^j}(p),
\end{aligned}$$

and (g^{ij}) is the inverse of the matrix (g_{ij}).

The *divergence* (on M) of a C^1-smooth vector field V (not necessarily tangential) at $p \in M$ is defined as follows. Let $\{X_1, X_2, \dots, X_n, Y_{n+1}, \dots, Y_m\}$ be an orthonormal basis of $\mathbb{R}^m$ such that $\{X_1, X_2, \dots, X_n\}$ forms a basis of T_pM. We write

$$V = \sum_{i=1}^{n} v^i X_i + \sum_{i=n+1}^{m} v^i Y_i.$$

Then

$$\operatorname{div}^M V(p) = \sum_{i=1}^{n} \langle \bar{\nabla}_{X_i} V, X_i \rangle = \sum_{i=1}^{n} \big\langle \big(\bar{\nabla}_{X_i} V\big)^{\top}, X_i \big\rangle.$$

Thus for a smooth vector field $V \in \mathcal{T}(M)$, $\operatorname{div}^M V(p)$ is the trace of the linear map $T_pM \to T_pM,\ v \mapsto \nabla_v V$. In local coordinates,

$$\operatorname{div}^M V = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \big(\sqrt{g} v^i\big),$$

where $g = \det(g_{ij})$. The *Laplacian* of a C^2-function $f \in C^2(M)$ is defined as

$$\Delta^M f = \operatorname{div}^M \nabla^M f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \big(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j}\big).$$

2.7 The second fundamental form of M

We denote by $\mathcal{N}(M)$ the set of all smooth mappings $V\colon M \to \mathbb{R}^m$ such that $V_p \in N_pM$ for all $p \in M$.

Definition 2.8. The *second fundamental form* of M is the map $\mathrm{I\!I}\colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{N}(M)$,

$$\mathrm{I\!I}(X, Y) = \Big(\bar{\nabla}_{\tilde{X}} \tilde{Y}\Big)^{\perp},$$

where $\tilde{X}$ and $\tilde{Y}$ are smooth extensions of X and Y, respectively. [$\mathrm{I\!I}$ reads as "two".]

Thus we have the *Gauss formula* on M:

$$\bar{\nabla}_X Y = \nabla_X Y + \mathrm{I\!I}(X, Y)$$

for vector fields $X, Y \in \mathcal{T}(M)$. Note again that the left hand side makes sense since $\big(\bar{\nabla}_X Y\big)_p$ depends only on $X_p \in T_pM$ and values of Y along any path $\gamma\colon]-\varepsilon, \varepsilon[\to M$, with $\gamma(0) = p$ and $\dot{\gamma}_0 = X_p$.

Lemma 2.9. *The second fundamental form is*

(a) independent of extensions of X and Y;

(b) symmetric in X and Y;

(c) C^∞-bilinear.

Proof. Let $\tilde{X}$ and $\tilde{Y}$ be some extensions of X and Y. Since $\bar{\nabla}$ is torsion-free, we have

$$\begin{aligned}\mathrm{II}(X,Y) - \mathrm{II}(Y,X) &= \left(\bar{\nabla}_{\tilde{X}}\tilde{Y}\right)^{\perp} - \left(\bar{\nabla}_{\tilde{Y}}\tilde{X}\right)^{\perp} \\ &= \left(\bar{\nabla}_{\tilde{X}}\tilde{Y} - \bar{\nabla}_{\tilde{Y}}\tilde{X}\right)^{\perp} \\ &= [\tilde{X},\tilde{Y}]^{\perp}.\end{aligned}$$

Since $X, Y \in \mathcal{T}(M)$, that is $\tilde{X}_p = X_p \in T_pM$ and $\tilde{Y}_p = Y_p \in T_pM$ at every point, also $[\tilde{X},\tilde{Y}]_p \in T_pM$. It follows that $[\tilde{X},\tilde{Y}]^{\perp} = 0$, and therefore II is symmetric. Since $\left(\bar{\nabla}_X Y\right)_p$ depends only on X_p (and values of Y along any path $\gamma\colon]-\varepsilon,\varepsilon[\to M$, with $\gamma(0) = p$ and $\dot{\gamma}_0 = X_p$), it is clear that $\mathrm{II}(X,Y)$ is independent of the extension chosen for X and that $\mathrm{II}(X,Y)$ is $C^\infty(M)$-linear in X. By symmetry, the same holds for Y. □

Lemma 2.10. [The Weingarten equation] *Suppose $X, Y \in \mathcal{T}(M)$ and $N \in \mathcal{N}(M)$. Then on M we have*

$$\langle \bar{\nabla}_X N, Y\rangle = -\langle N, \mathrm{II}(X,Y)\rangle,$$

where X, Y, and N are extended to $\mathbb{R}^m$ (and still denoted by X, Y, N).

Proof. Since $Y \in \mathcal{T}(M)$ and $N \in \mathcal{N}(M)$, we have $\langle N, Y\rangle \equiv 0$ on M. Furthermore, since $X \in \mathcal{T}(M)$, we have on M

$$\begin{aligned}0 = X\langle N, Y\rangle &= \langle \bar{\nabla}_X N, Y\rangle + \langle N, \bar{\nabla}_X Y\rangle \\ &= \langle \bar{\nabla}_X N, Y\rangle + \langle N, \nabla_X Y + \mathrm{II}(X,Y)\rangle \\ &= \langle \bar{\nabla}_X N, Y\rangle + \langle N, \mathrm{II}(X,Y)\rangle.\end{aligned}$$

□

As a *geometric interpretation* we note that the second fundamental form $\mathrm{II}_p(V,V)$ is the Euclidean acceleration $\gamma''_{\mathrm{Eucl}}(0)$ of the geodesic on M with the initial velocity vector $V_p = \dot{\gamma}_0$ at $p \in M$. An explanation for this is the *Gauss formula along a smooth path* $\gamma\colon I \to \mathbb{R}^m$. More precisely, let $V\colon I \to \mathbb{R}^m$ be a smooth vector field along γ, i.e. $V_t \in T_{\gamma(t)}\mathbb{R}^m$ for all $t \in I$. Then

$$\bar{D}_t V = D_t V + \mathrm{II}(\dot{\gamma}, V),$$

where $\bar{D}_t V = V'_t \in \mathbb{R}^m$. Now, if $V = \dot{\gamma}$, then

$$\bar{D}_t\dot{\gamma} = D_t\dot{\gamma} + \mathrm{II}(\dot{\gamma},\dot{\gamma}),$$

and if γ is a geodesic on M (i.e. $D_t\dot{\gamma} = 0$), we further have

$$\bar{D}_t\dot{\gamma} = \mathrm{II}(\dot{\gamma},\dot{\gamma}).$$

Definition 2.11. The *mean curvature vector* H on M is ("the trace of the second fundamental form")

$$H = \sum_{i=1}^{n} \mathrm{II}(X_i, X_i),$$

where $X_1, \ldots, X_n$ is an orthonormal basis of T_pM.

Note that $H_p \in N_pM$. *Exercise:* If $v_1, v_2, \ldots, v_n$ is an arbitrary basis of T_pM and $g_{ij} = \langle v_i, v_j\rangle$, then

$$H_p = \sum_{i,j=1}^{n} g^{ij}\,\mathrm{II}_p(v_i, v_j),$$

where (g^{ij}) is the inverse of the matrix (g_{ij}).

2.12 Hypersurfaces of $\mathbb{R}^m$, the scalar second fundamental form, the Weingarten map, and the shape operator

Let M be an $(m-1)$-dimensional submanifold of $\mathbb{R}^m$, i.e. a *hypersurface*.

Definition 2.13. The *scalar second fundamental form* of M is the symmetric 2-tensor defined by

$$h(X,Y) = \langle \mathrm{II}(X,Y), N\rangle,$$

where $N \in \mathcal{N}(M)$ is a smooth unit normal vector field.

Since M is of co-dimension 1, the unit normal vector N_p spans N_pM at every point $p \in M$. Hence

$$\mathrm{II}(X,Y) = h(X,Y)N.$$

Note that the sign of h depends on the choice of N (versus $-N$). We have the *Gauss formula* for hypersurfaces of $\mathbb{R}^m$:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)N.$$

Definition 2.14. The *Weingarten map* $L\colon TM \to TM$ is defined as

$$LX = -\bar{\nabla}_X N.$$

Remarks 2.15.

1. Usually the Weingarten map is defined as $LX = \bar{\nabla}_X N$. However, this is just a matter of convention since changing the sign of N changes the sign of L as well. The Weingarten map is also called the *shape operator*.
2. The target space is indeed TM. In fact, for each $p \in M$, the Weingarten map is a self-adjoint endomorphism of T_pM; see Lemma 2.16

Lemma 2.16. *For each $p \in M$, the Weingarten map is a self-adjoint endomorphism of T_pM.*

Proof. First we prove that, for each $p \in M$, the target space of L is T_pM. Let $X_p \in T_pM$ be arbitrary. Since $\langle N, N\rangle \equiv 1$, we have

$$0 = X_p\langle N,N\rangle = 2\langle (\bar{\nabla}_X N)_p, N_p\rangle.$$

Hence $LX_p = -(\bar{\nabla}_X N)_p \in T_pM$. Clearly L is linear. To prove that it is self-adjoint, let $p \in M$ and $v, w \in T_pM$. By using the Weingarten equation and the symmetry of II_p, we obtain

$$\begin{aligned}\langle Lv, w\rangle - \langle v, Lw\rangle &= -\langle \bar{\nabla}_v N, w\rangle + \langle v, \bar{\nabla}_w N\rangle \\ &= \langle N, \mathrm{II}_p(v,w)\rangle - \langle N, \mathrm{II}(w,v)\rangle \\ &= 0.\end{aligned}$$

Hence L is self-adjoint. □

Remarks 2.17. *On the terminology and notation:*
The first fundamental form is just the restriction to TM of the standard inner product of $\mathbb{R}^m$:

$$I(v,w) = \langle v, w\rangle.$$

The scalar second fundamental form is

$$\begin{aligned}h(v,w) &= \langle \mathrm{II}(v,w), N\rangle = -\langle \bar{\nabla}_v N, w\rangle \\ &= \langle Lv, w\rangle.\end{aligned}$$

In literature, it is sometimes called the second fundamental form and denoted by II.
The third fundamental form is

$$\mathrm{III}(v,w) = \langle L^2 v, w\rangle = \langle Lv, Lw\rangle.$$

Since for every $p \in M$, $L\colon T_pM \to T_pM$ is self-adjoint, it follows from linear algebra that it has real eigenvalues $\kappa_1, \kappa_2, \ldots, \kappa_{m-1}$ and that there exists an orthonormal basis $E_1, E_2, \ldots, E_{m-1}$ of T_pM consisting of eigenvectors: $LE_i = \kappa_i E_i$, $i = 1, \ldots, m-1$. The eigenvalues of L are called the *principal curvatures* and the corresponding eigenvectors are called *principal directions*.

If vectors X and Y are given in the orthonormal basis as

$$X = \sum_{i=1}^{m-1} x^i E_i \quad \text{and} \quad Y = \sum_{j=1}^{m-1} y^j E_j,$$

then the scalar second fundamental form has a simple expression

$$\begin{aligned} h(X,Y) &= \left\langle L\sum_{i=1}^{m-1} x^i E_i, \sum_{j=1}^{m-1} y^j E_j \right\rangle \\ &= \sum_{i=1}^{m-1} \kappa_i x^i y^i \end{aligned}$$

Let us return to the general case of an n-dimensional submanifold $M \subset \mathbb{R}^m$, with co-dimension $k = m - n \geq 1$. For each $p \in M$ and a (unit) vector $\nu \in N_pM$, we define a symmetric bi-linear form $h_\nu\colon T_pM \times T_pM \to \mathbb{R}$ by

$$h_\nu(v,w) = \langle \mathrm{II}_p(v,w), \nu\rangle$$

and a self-adjoint linear map $S_\nu\colon T_pM \to T_pM$ such that

$$\langle S_\nu(v), w\rangle = h_\nu(v,w)$$

for all $w \in T_pM$. The mapping h_ν could be called the *scalar second fundamental form along* ν and S_ν could be called the *shape operator along* ν.

Lemma 2.18. *Let $p \in M$, $v \in T_pM$, and $\nu \in N_pM$. Furthermore, let N be a local smooth extension of ν to a neighborhood $U \subset M$ of p such that $N \in \mathcal{N}(U)$, i.e. $N_q \in N_qM$ $\forall q \in U$. Then*

$$S_\nu(v) = -\left(\bar{\nabla}_v N\right)^\top = -\nabla_v N.$$

Proof. Let $w \in T_pM$ and let $V, W \in \mathcal{T}(U)$ be smooth extensions of v and w, respectively. Then $\langle N, W\rangle \equiv 0$, and therefore

$$\langle \bar{\nabla}_V N, W\rangle + \langle N, \bar{\nabla}_V W\rangle = V\langle N, W\rangle = 0.$$

It follows that

$$\begin{aligned} \langle S_\nu(v), w\rangle &= \langle \mathrm{II}_p(v,w), \nu\rangle = \langle \mathrm{II}(V,W), N\rangle_p \\ &= \langle \bar{\nabla}_V W - \nabla_V W, N\rangle_p = \langle \bar{\nabla}_V W, N\rangle_p - \underbrace{\langle \nabla_V W, N\rangle_p}_{=0} \\ &= \langle \bar{\nabla}_V W, N\rangle_p \\ &= -\langle \bar{\nabla}_V N, W\rangle_p. \end{aligned}$$

Hence

$$\langle S_\nu(v), w\rangle = -\langle \bar{\nabla}_v N, w\rangle \quad \forall w \in T_pM,$$

and so

$$S_\nu(v) = -\left(\bar{\nabla}_v N\right)^\top$$

□

Let $p \in M$ and $\nu \in N_pM$, $|\nu| = 1$. Since $S_\nu \colon T_pM \to T_pM$ is self-adjoint, there exists an orthonormal basis $E_1, E_2, \ldots, E_n$ of T_pM consisting of eigenvectors associated with real eigenvalues $\kappa_1, \ldots, \kappa_2$ of S_ν $(S_\nu E_i = \kappa_i E_i)$. I do not know whether these eigenvalues are called principal curvatures in co-dimensions $k > 1$.

2.19 Curvatures on M

Let $M \subset \mathbb{R}^m$ be a smooth hypersurface and let $p \in M$. Since the determinant and trace of the Weingarten map $L \colon T_pM \to T_pM$ are basis-independent, there are two combinations of eigenvalues of L that are well-defined and geometrically significant.

The *Gaussian curvature* of M at p is the determinant

$$K = \det L = \kappa_1\kappa_2\cdots\kappa_{m-1}$$

and the *mean curvature* of M at p is the mean of the trace

$$H = \frac{1}{m-1}\operatorname{tr} L = \frac{1}{m-1}\big(\kappa_1 + \kappa_2 + \cdots + \kappa_{m-1}\big).$$

Clearly the mean curvature changes its sign if the normal vector field is changed (from N to $-N$). Similarly, if $m-1$ is odd, the sign of the Gaussian curvature changes, whereas the Gaussian curvature is independent of the choice of (the direction of) N if $m-1$ is even, in particular, if M is a hypersurface of $\mathbb{R}^3$.

Recall the definition of the Riemannian curvature tensor $R \colon \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$,

$$R^M(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Note that the Riemannian curvature tensor $\bar{R}$ of $\mathbb{R}^m$ vanishes identically.

The sectional curvature of a 2-dimensional subspace $P \subset T_pM$ spanned by vectors $v, w \in T_pM$ is defined by

$$K^M(P) = \frac{\langle R^M(v,w)w, v\rangle}{|v \wedge w|^2},$$

where

$$|v \wedge w| = \sqrt{|v|^2|w|^2 - \langle v, w\rangle^2}$$

is the area of the parallelogram spanned by v and w. Note that $K^M(P)$ is independent of the choice of linearly independent vectors $v, w \in P$. In general, sectional curvatures on a submanifold of a Riemannian manifold are given in terms of sectional curvatures of the ambient space and the second fundamental form. In our setting of a submanifold M of R^m this so-called *Gauss equation* reads as follows:

$$K^M(P)|v \wedge w|^2 - \underbrace{\bar{K}(P)|v \wedge w|^2}_{=0} = \langle \mathrm{I\!I}_p(v,v), \mathrm{I\!I}_p(w,w)\rangle - |\,\mathrm{I\!I}_p(v,w)|^2.$$

Here $\bar{K}(P)$ denotes the sectional curvature of P with respect to the ambient space which in our setting is $\mathbb{R}^m$ and therefore $\bar{K} \equiv 0$. In particular, for orthonormal vectors $v, w \in P$ we have

$$K(P) = \langle \mathrm{II}_p(v,v), \mathrm{II}_p(w,w) \rangle - |\, \mathrm{II}_p(v,w)|^2$$

since $K(P)$ does not depend on $v, w \in P$ provided $P = \mathrm{span}(v,w)$.

Let then $M \subset \mathbb{R}^m$ be a smooth hypersurface. Let $E_1, E_2, \ldots, E_{m-1}$ be an orthonormal basis of T_pM, $p \in M$, consisting of the eigenvectors of L with eigenvalues $\kappa_1, \ldots, \kappa_{m-1}$. Then

$$\mathrm{II}_p(E_i, E_j) = h(E_i, E_j)N,$$

where $N \in N_pM$ is a unit vector and

$$\begin{aligned} h(E_i, E_j) &= \langle \mathrm{II}(E_i, E_j), N \rangle = -\langle \bar{\nabla}_{E_i} N, E_j \rangle \\ &= \langle L E_i, E_j \rangle = \kappa_i \langle E_i, E_j \rangle \\ &= \kappa_i \delta_{ij}. \end{aligned}$$

Hence $\mathrm{II}_p(E_i, E_j) = \kappa_i \delta_{ij} N$ and therefore

$$K(P) = \langle \mathrm{II}_p(E_i, E_i), \mathrm{II}_p(E_j, E_j) \rangle - \underbrace{|\, \mathrm{II}_p(E_i, E_j)|^2}_{=0} = \kappa_i \kappa_j$$

for a 2-dimensional subspace $P = \mathrm{span}(E_i, E_j) \subset T_pM$.

3 First variation formula and some of its consequences

3.1 Mean curvature and the Laplacian

We start with recalling the following *Jacobi formula* for the derivative of a determinant whose proof is left as an exercise.

Lemma 3.2. *Let $a_{ij} \colon \mathbb{R}^d \to \mathbb{R}$ be smooth functions, with $i, j = 1, \ldots, n$, and let $A = (a_{ij})$. Then in the open set $\{x \in \mathbb{R}^m \colon \det A \neq 0\}$ we have*

$$\frac{\partial}{\partial x^\ell} \log \det A = \mathrm{tr} \left(\frac{\partial A}{\partial x^\ell} A^{-1} \right)$$

for $\ell = 1, \ldots, d$.

Writing $A^{-1} = (a^{ij})$, the right hand side reads as

$$\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x^\ell} a^{ji},$$

and so

$$\frac{\partial \det A}{\partial x^\ell} = \det A \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x^\ell} a^{ji}. \tag{3.3}$$

Suppose that $M \subset \mathbb{R}^m$ is a smooth n-dimensional submanifold and let $\varphi\colon U \to \Omega \subset R^n$ be a chart defined in an open set $U \subset M$. Furthermore, let $F = \varphi^{-1}\colon \Omega \to U$ be local parametrization. As in 2.6 F induces a frame $\{\frac{\partial F}{\partial x^j}\}$,

$$\left(\frac{\partial F}{\partial x^j}\right)_p = \left(\frac{\partial F_1}{\partial x^j}(\varphi(p)), \dots, \frac{\partial F_m}{\partial x^j}(\varphi(p))\right) \in T_pM,$$

on U. Now

$$\bar\nabla_{\frac{\partial F}{\partial x^i}} \tfrac{\partial F}{\partial x^j} = \tfrac{\partial^2 F}{\partial x^i \partial x^j},$$

$$\left(\bar\nabla_{\frac{\partial F}{\partial x^i}} \tfrac{\partial F}{\partial x^j}\right)_p = \left(\frac{\partial^2 F_1}{\partial x^i \partial x^j}, \dots, \frac{\partial^2 F_m}{\partial x^i \partial x^j}\right)(\varphi(p)) \in \mathbb{R}^m.$$

Hence the mean curvature vector H_p at $p \in U$ is given by

$$\begin{aligned} H_p &= \sum_{i,j=1}^n g^{ij}(p)\, \mathrm{II}_p\big(\tfrac{\partial F}{\partial x^i}, \tfrac{\partial F}{\partial x^j}\big) \\ &= \sum_{i,j=1}^n g^{ij}(p) \left(\bar\nabla_{\frac{\partial F}{\partial x^i}} \tfrac{\partial F}{\partial x^j}\right)_p^\perp \\ &= \left(\sum_{i,j=1}^n g^{ij}(p) \frac{\partial^2 F}{\partial x^i \partial x^j}(\varphi(p))\right)^\perp . \end{aligned}$$

Next we express the mean curvature vector as the Laplacian (on M) of the inclusion $\pi\colon M \hookrightarrow \mathbb{R}^m$.

Theorem 3.4. *Suppose that $M \subset \mathbb{R}^m$ is a smooth n-dimensional submanifold and let $\pi\colon M \hookrightarrow \mathbb{R}^m$, $\pi = (\pi_1, \dots, \pi_m)$, be the inclusion. Then*

$$H_p = \Delta^M \pi(p) = \big(\Delta^M \pi_1, \dots, \Delta^M \pi_m\big)(p)$$

for $p \in M$.

Proof. Fix $p \in M$ and let $\varphi\colon U \to \Omega \subset \mathbb{R}^n$ be a chart at p and

$$\frac{\partial}{\partial x^i}, \quad i = 1, \dots, n,$$

the coordinate frame associated to the chart (U, φ). Furthermore, let $F = \varphi^{-1}\colon \Omega \to U$ be the corresponding (local) parametrization. Then, in fact,

$$\left(\frac{\partial}{\partial x^j}\right)_p \pi_i = \frac{\partial}{\partial x^j}\big(\underbrace{\pi_i \circ \varphi^{-1}}_{=\pi_i \circ F = F_i}\big)(\varphi(p)) = \frac{\partial F_i}{\partial x^j}(\varphi(p)).$$

First we claim that $\Delta^M \pi(p) \in N_pM$, that is

$$\Delta^M \pi(p) \cdot \tfrac{\partial F}{\partial x^k} = \Delta^M \pi(p) \cdot \tfrac{\partial \pi}{\partial x^k} = 0$$

for all $k = 1, \ldots, n$. We compute by using (3.3) and the symmetry of (g_{ij})

$$
\begin{aligned}
\Delta^M \pi(p) \cdot \tfrac{\partial \pi}{\partial x^k} &= \Big(\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \big(\sqrt{g} g^{ij} \tfrac{\partial \pi}{\partial x^j} \big) \Big) \cdot \frac{\partial \pi}{\partial x^k} \\
&= \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \Big(\sqrt{g} g^{ij} \underbrace{\frac{\partial \pi}{\partial x^j} \cdot \frac{\partial \pi}{\partial x^k}}_{=g_{jk}} \Big) - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \Big(\sqrt{g}\, \underbrace{g^{ij} g_{jk}}_{\delta_{ik}} \Big) - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= \frac{1}{\sqrt{g}} \frac{1}{2\sqrt{g}} \frac{\partial g}{\partial x^k} - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} \frac{\partial}{\partial x^k} \Big\langle \frac{\partial \pi}{\partial x^i}, \frac{\partial \pi}{\partial x^j} \Big\rangle - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} \left(\frac{\partial^2 \pi}{\partial x^i \partial x^k} \cdot \frac{\partial \pi}{\partial x^j} + \frac{\partial^2 \pi}{\partial x^j \partial x^k} \cdot \frac{\partial \pi}{\partial x^i} \right) - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= 0.
\end{aligned}
$$

Thus $\Delta^M \pi(p) \in N_p M$ since $\big(\frac{\partial \pi}{\partial x^k} \big)_p$, $k = 1, \ldots, n$, forms a basis of $T_p M$. Furthermore,

$$
\begin{aligned}
\Delta^M \pi(p) &= \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \Big(\sqrt{g} g^{ij} \frac{\partial \pi}{\partial x^j} \Big) \\
&= \underbrace{\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \Big(\sqrt{g} g^{ij} \Big) \frac{\partial \pi}{\partial x^j}}_{\in T_p M} + \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 \pi}{\partial x^i \partial x^j}.
\end{aligned}
$$

On the other hand, since $\Delta^M \pi(p) \in N_p M$, we have

$$
\begin{aligned}
\Delta^M \pi(p) &= \big(\Delta^M \pi(p) \big)^\perp \\
&= \underbrace{\Big(\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \Big(\sqrt{g} g^{ij} \Big) \frac{\partial \pi}{\partial x^j} \Big)^\perp}_{=0} + \Big(\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 \pi}{\partial x^i \partial x^j} \Big)^\perp \\
&= \Big(\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 \pi}{\partial x^i \partial x^j} \Big)^\perp = H_p
\end{aligned}
$$

as claimed. □

3.5 First variation formula

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \colon \Omega \to \mathbb{R}^m$ an immersion. Denote $M = f\Omega$. Since every $x \in \Omega$ has a neighborhood $U \subset \Omega$ such that $f|\Omega$ is an embedding, we can define the "tangent space" $T_{f(x)} M$

and the normal space $N_{f(x)}M$ as $T_{f(x)}M = T_{f(x)}U = df(x)\mathbb{R}^n$ and $N_{f(x)}M = N_{f(x)}U$ although f need not be injective. Furthermore, let $\varphi \in C_0^\infty(\Omega)$ be a smooth real-valued function with compact support in Ω and let $N\colon \Omega \to \mathbb{S}^{m-1}$ be a smooth mapping such that $N_x = N(x) \in N_{f(x)}M$ for every $x \in \Omega$. We define a variation of M (more precisely, a variation of the immersion $f\colon \Omega \to \mathbb{R}^m$) with compact support as a mapping

$$F\colon \Omega\times]-\varepsilon,\varepsilon[\to \mathbb{R}^m, \quad F(x,t) = f(x) + t\varphi(x)N_x.$$

Here $\varepsilon > 0$ is so small that $F\colon \tilde\Omega\times]-\varepsilon,\varepsilon[\to \mathbb{R}^m$ is an immersion, where $\tilde\Omega = \{x \in \Omega\colon \varphi(x) \neq 0\}$. To prove that such ε exists, we have to show that $dF(x,t)$ is injective if $x \in \tilde\Omega$ and $|t| < \varepsilon$. To that end, we write the matrix of $dF(x,t)$ in the standard bases of $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and $\mathbb{R}^m$ as the $m \times (n+1)$-matrix

$$\begin{pmatrix} df(x) & 0 \end{pmatrix} + \begin{pmatrix} td\big(\varphi(x)N_x\big) & \varphi(x)N_x \end{pmatrix}.$$

Above $df(x)$ and $td\big(\varphi(x)N_x\big)$ are $m \times n$-matrices and 0 and $\varphi(x)N_x$ are $m \times 1$-matrices (columns). Since $\varphi \in C_0^\infty(\Omega)$ and f is an immersion, we have

$$\begin{aligned} m_1 &= \inf_{x\in\tilde\Omega}\{\min|df(x)v|\colon |v| = 1\} > 0 \quad \text{and} \\ m_2 &= \sup_{x\in\tilde\Omega}\{\max|d\big(\varphi(x)N_x\big)v|\colon |v| = 1\} < \infty. \end{aligned}$$

Thus we may choose $0 < \varepsilon < m_1/m_2$. Suppose that the vector $w = (w_x, w_t) \in \mathbb{R}^n \times \mathbb{R}$ belongs to the kernel of $dF(x,t)$, with $(x,t) \in \tilde\Omega\times]-\varepsilon,\varepsilon[$. Then

$$\begin{aligned} 0 &= df(x)w_x + td\big(\varphi(x)N_x\big)w_x + \varphi(x)w_tN_x \\ &= \big(df(x) + td\big(\varphi(x)N_x\big)\big)\, w_x + \varphi(x)w_tN_x. \end{aligned}$$

Now, if $|t| < \varepsilon$ and $w_x \neq 0$, we would have

$$\Big(\underbrace{df(x)w_x + td\big(\varphi(x)N_x\big)w_x}_{\in T_{f(x)}M}\Big)^\top \neq 0$$

which is impossible since $\varphi(x)w_tN_x \in N_{f(x)}M$. It follows that $w_x = 0$. Since $\varphi(x) \neq 0$ in $\tilde\Omega$, we also have $w_t = 0$. Hence $dF(x,t)$ is injective. Denote by $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, \partial_t\}$ the standard basis of $\mathbb{R}^{n+1}$ and define vector fields F_{x_i} and F_t along F by setting

$$F_{x_i}(x,t) = dF(x,t)\partial_{x_i} \quad \text{and} \quad F_t(x,t) = dF(x,t)\partial_t.$$

Then F_{x_i} and F_t commute because

$$[F_{x_i}, F_t] = dF\,\underbrace{[\partial_{x_i}, \partial_t]}_{=0} = 0.$$

Note that $F_t(x,0) = dF(x,0)\partial_t = \varphi(x)N_x \in N_{f(x)}M$. We define

$$g_{ij}(x,t) = \langle F_{x_i}(x,t), F_{x_j}(x,t)\rangle$$

and $g(x,t) = \det g_{ij}(x,t)$. The volume of $M_t = F(\Omega,t)$ is given by

$$\operatorname{Vol} M_t = \int_\Omega \sqrt{g(x,t)}dx.$$

Hence

$$
\begin{aligned}
\frac{d}{dt}\operatorname{Vol} M_{t|t=0} &= \int_\Omega \frac{\partial}{\partial t}\sqrt{g(x,t)}_{|t=0}dx \\
&= \frac{1}{2}\int_\Omega \frac{1}{\sqrt{g(x,0)}}\frac{\partial}{\partial t}g(x,t)_{|t=0}dx \\
&= \frac{1}{2}\int_\Omega \sqrt{g(x,0)}\sum_{i,j=1}^n g^{ij}(x,0)\frac{\partial g_{ij}(x,t)}{\partial t}_{|t=0}dx \\
&= \frac{1}{2}\int_\Omega \sqrt{g(x,0)}\sum_{i,j=1}^n g^{ij}(x,0)\frac{\partial}{\partial t}\langle F_{x_i},F_{x_j}\rangle(x,t)_{|t=0}dx \\
&= \frac{1}{2}\int_\Omega \sqrt{g(x,0)}\sum_{i,j=1}^n g^{ij}(x,0)\left(\big\langle\bar\nabla_{F_t}F_{x_i},F_{x_j}\big\rangle+\big\langle\bar\nabla_{F_t}F_{x_j},F_{x_i}\big\rangle\right)(x,0)dx \\
&= \int_\Omega \sqrt{g(x,0)}\sum_{i,j=1}^n g^{ij}(x,0)\big\langle\bar\nabla_{F_t}F_{x_i},F_{x_j}\big\rangle(x,0)dx \\
&= \int_\Omega \sqrt{g(x,0)}\sum_{i,j=1}^n g^{ij}(x,0)\big\langle\bar\nabla_{F_{x_i}}F_t,F_{x_j}\big\rangle(x,0)dx \\
&= \int_\Omega \sqrt{g(x,0)}\sum_{i,j=1}^n g^{ij}(x,0)\big\langle\bar\nabla_{F_{x_i}}\big(\varphi(x)N_x\big),F_{x_j}\big\rangle(x,0)dx \\
&= -\int_\Omega \sqrt{g(x,0)}\sum_{i,j=1}^n g^{ij}(x,0)\big\langle\mathrm{II}\big(F_{x_i},F_{x_j}\big),\varphi(x)N_x\big\rangle(x,0)dx \\
&= -\int_\Omega \sqrt{g(x,0)}\big\langle H_{f(x)},\varphi(x)N_x\big\rangle(x,0)dx \\
&= -\int_M \langle H,V\rangle.
\end{aligned}
$$

Above $H_{f(x)}$ denotes the mean curvature vector at $f(x)$ of fU, where U is a sufficiently small neighborhood of x so that $f|U$ is an embedding. Moreover, the last expression

$$-\int_M \langle H,V\rangle$$

should be interpreted as a shorthand notation in case the immersion $f\colon \Omega\to\mathbb{R}^m$ is non- injective, whereas $V_p=\varphi(f^{-1}(p))N_{f^{-1}(p)}$ for an injective immersion f.

Above we considered a "normal" variation with compact support, i.e. $N_x\in\mathbb{S}^{m-1}$ is assumed to be normal to M at $f(x)$; $N_x\in N_{f(x)}M$. For a general case, we decompose an arbitrary smooth mapping $X\colon\Omega\to\mathbb{S}^{m-1}$ as

$$X=X^\top+X^\perp,$$

where $X_x^\top\in T_{f(x)}M$ and $X^\perp\in N_{f(x)}M$. Then a variation in direction X is $F\colon\Omega\times]-\varepsilon,\varepsilon[\to\mathbb{R}^m$,

$$F(x,t)=f(x)+t\varphi(x)X_x=f(x)+t\varphi(x)X_x^\top+t\varphi(x)X_x^\perp.$$

The tangential variation $f(x)+t\varphi(x)X_x^\top$ preserves the volume of M, and therefore the general case reduces to a normal variation. We get the following characterization from the first variation formula.

Theorem 3.6. *An immersed submanifold $M \subset \mathbb{R}^m$ is a critical point of the volume functional for all compactly supported variations if and only if the mean curvature vector vanishes identically; $H \equiv 0$.*

Proof. If $H \equiv 0$, then M is clearly a critical point for all compactly supported variations. Conversely, supposing the M is a critical point, the above shows that

$$\int_M \langle H, V \rangle = 0$$

for all smooth $V \colon M \to \mathbb{R}^m$ with compact support. In particular, this holds for $V = \varphi H$ for every non-negative $\varphi \in C_0^\infty(M)$. Hence $H \equiv 0$. □

Definition 3.7. An immersed submanifold $M \subset \mathbb{R}^m$ is *minimal* if $H \equiv 0$ on M.

Remark 3.8. Let $X \colon M \to \mathbb{R}^m$ be smooth and compactly supported. It follows from the proof of the first variation formula that

$$\frac{d}{dt} \operatorname{Vol}(M_t)_{|t=0} = \int_\Omega \sqrt{g(x,0)} \sum_{i,j=1}^n g^{ij}(x,0) \big\langle \bar{\nabla}_{F_{x_i}} X, F_{x_j} \big\rangle (x,0) dx = \int_M \operatorname{div}^M X$$

for a variation in direction X.

Lemma 3.9. *An immersed submanifold M is minimal if and only if*

$$\int_M \operatorname{div}^M X = 0$$

for all smooth (not necessary tangential) $X \colon M \to \mathbb{R}^m$ with compact support.

Theorem 3.10. *Let $M^n \subset \mathbb{R}^m$ be a smooth manifold and let $\pi = (\pi_1, \dots, \pi_m) \colon M \hookrightarrow \mathbb{R}^m$ be the inclusion. Then M is minimal if and only if each coordinate function $\pi_i \colon M \to \mathbb{R}$ is harmonic.*

Proof. This follows directly from Theorem 3.4 which states that $\Delta^M \pi = H$. □

The result above could be proved also by applying a weak formulation of harmonicity. For that purpose, let $\eta \in C_0^\infty(M)$ be a real-valued smooth function with compact support and let E_i be the (constant) i-th coordinate vector field on $\mathbb{R}^m$,

$$E_i = (0, 0, \dots, 0, \overset{i}{1}, 0, \dots, 0).$$

Then $\bar{\nabla}_X E_i \equiv 0$ for every vector field X. Furthermore, ηE_i is a smooth (not necessary tangential) vector field on M with compact support and

$$\operatorname{div}^M(\eta E_i) = \eta \operatorname{div}^M E_i + \langle \nabla^M \eta, E_i \rangle.$$

By the definition of the divergence div^M we obtain

$$\operatorname{div}^M E_j = \sum_{j=1}^n \big\langle \bar{\nabla}_{X_j} E_i, X_j \big\rangle = 0,$$

where $\{X_1, \dots, X_n\}$ forms a basis of T_pM. Hence

$$\int_M \operatorname{div}^M(\eta E_i) = \int_M \langle \nabla^M \eta, E_i \rangle = \int_M \langle \nabla^M \eta, \nabla^M \pi_i \rangle.$$

Since M is minimal, we have

$$\int_M \operatorname{div}^M(\eta E_i) = 0,$$

and so

$$\int_M \langle \nabla^M \eta, \nabla^M \pi \rangle = 0 \tag{3.11}$$

for all $i = 1, \dots, m$ and $\eta \in C_0^\infty(M)$. That is, each π_i is harmonic (in the weak sense). Conversely, if each π_i is harmonic, then (3.11) holds for all $i = 1, \dots, m$ and $\eta \in C_0^\infty(M)$. Let

$$X = \sum_{i=1}^m \eta_i E_i$$

be an arbitrary smooth vector field with compact support. Then

$$\int_M \operatorname{div}^M X = \sum_{i=1}^m \int_M \operatorname{div}^M(\eta_i E_i) = \sum_{i=1}^m \int_M \langle \nabla^M \eta_i, \nabla^M \pi_i \rangle = 0.$$

3.12 Some consequences

Recall that the *convex hull* of a compact set $K \subset \mathbb{R}^m$ is the smallest convex set, denoted by $\operatorname{Conv}(K)$, containing K. It is the intersection of all closed half-spaces containing K.

Corollary 3.13. *Let $M^n \subset \mathbb{R}^m$ be a minimal surface such that $\bar{M} = M \cup \partial M$ is compact. Then $\bar{M} \subset \operatorname{Conv}(\partial M)$.*

Proof. This follows from the harmonicity of π_i and the maximum principle for harmonic functions. Every closed half-space $H \subset \mathbb{R}^m$ can be written in a form

$$H = \{x \in \mathbb{R}^m \colon x \cdot e \le a\}$$

for some $e \in \mathbb{S}^{m-1}$ and $a \in \mathbb{R}$. Denote it by $H = H(e, a)$. For each $e \in \mathbb{S}^{m-1}$ and $a \in \mathbb{R}$, the function

$$u(x) = \langle x, e \rangle = \sum_{i=1}^m e_i x_i$$

is harmonic on M since

$$\Delta^m u = \sum_{i=1}^m e_i \Delta^M \pi_i = 0.$$

By the maximum principle for harmonic functions

$$u|M \le \max u|\partial M =: m_e.$$

It follows that

$$\operatorname{Conv}(\partial M) = \bigcap_{e \in \mathbb{S}^{m-1}} H(e, m_e)$$

and

$$u(x) = \langle x, e \rangle \le m_e \ \forall x \in M,$$

so

$$\bar{M} \subset H(e, m_e) \ \forall e \in \mathbb{S}^{m-1}.$$

Hence

$$M \subset \mathrm{Conv}(\partial M).$$

□

Corollary 3.14. *Suppose that $M^2 \subset \mathbb{R}^m$ is a minimal surface which is topologically a disk and that $\bar{M} = M \cup \partial M$ is compact. Suppose that $K \subset \mathbb{R}^m$ is a compact convex set such that $K \cap \partial M = \emptyset$. Then each component of $K \cap M$ is simply connected.*

Proof. Suppose that there exists a component of $K \cap M$ that is not simply connected. Then there exits a Jordan curve γ that does not bound a topogical disk in $K \cap M$. On the other hand, since M is simply connected, γ bounds a topological disk $D \subset M$. Now $\partial D = \gamma \subset K$ but $D \not\subset K$. This is a contradiction since $\mathrm{Conv}(\partial D) \subset K$ and, as a minimal surface, $D \subset \mathrm{Conv}(\partial D)$. □

4 Bernstein's theorem

4.1 The Gauss map

Let $M \subset \mathbb{R}^m$ be an oriented smooth hypersurface, i.e. an $(m-1)$-dimensional submanifold. The *Gauss map* is a continuous (choice of a) unit normal vector field $N\colon M \to \mathbb{S}^{m-1}$, $N_x \in N_xM$. The differential of N at $x \in M$,

$$dN(x)\colon T_xM \to T_{N_x}\mathbb{S}^{m-1},$$

can be identified with $-$ the Weingarten map at x,

$$-L\colon T_xM \to T_xM,$$

as follows. Let $E_1, \ldots, E_{m-1}$ be a local orthonormal frame on M. Since $E_i \perp N\ \forall i$ and N_x itself is normal to $\mathbb{S}^{m-1}$ at $N_x \in \mathbb{S}^{m-1}$, vector fields $E_1, \ldots, E_{m-1}$ form an orthonormal frame on the image $N(M)$ as well. Then the matrix of dN with respect to $\{E_i\}_{i=1}^{m-1}$ is given by

$$\begin{aligned}(dN)_{ij} &= \big(\langle \bar\nabla_{E_j} N, E_i\rangle\big) = -\big(\langle N, \bar\nabla_{E_j} E_i\rangle\big)\\ &= -\big(\langle N, \mathrm{II}(E_j, E_i)\rangle\big) = -\big(h(E_j, E_i)\big),\end{aligned}$$

where h is the scalar second fundamental form. On the other hand, $LX = -\bar\nabla_X N$, and so

$$(dN)_{ij} = -\big(\langle LE_j, E_i\rangle\big) = -(L)_{ij}.$$

Let us consider next the case $m = 3$ and suppose that $M^2 \subset \mathbb{R}^3$ is minimal. Since the mean curvature vanishes identically, the eigenvalues of $dN = -L$ are $\kappa \geq 0$ and $-\kappa$. Hence the matrix of dN in the orthonormal basis formed by the eigenvectors is the diagonal matrix

$$\begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}.$$

The determinant is $\det dN = -\kappa^2 = K$ (the Gauss curvature) and the (operator) norm of dN is $|dN| = \kappa$. Thus the Gauss map is (anti)conformal ($|dN|^2 = -\det dN$). Combining the Gauss map $N\colon M \to \mathbb{S}^2$ with the stereographic projection $P\colon \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ we obtain that $P \circ N\colon M \to \mathbb{C} \cup \{\infty\}$ is (anti)meromorphic.

4.2 Mean curvature and Gauss curvature of a graph

For notational simplicity let us consider the case $m = 3$. Suppose that $M \subset \mathbb{R}^3$ is the graph of a smooth function $u\colon \Omega \to \mathbb{R}$, with $\Omega \subset \mathbb{R}^2$ open. Then M is parametrized by $F\colon \Omega \to M$, $F(x,y) = \big(x, y, u(x,y)\big)$. The vector fields

$$X_1 = \partial_x F = (1, 0, u_x) \quad \text{and} \quad X_2 = \partial_y F = (0, 1, u_y)$$

form a frame on M. The components of the metric are given by

$$g_{11} = X_1 \cdot X_1 = 1 + u_x^2, \quad g_{22} = X_2 \cdot X_2 = 1 + u_y^2, \quad \text{and} \quad g_{12} = g_{21} = u_x u_y.$$

Hence

$$(g_{ij}) = \begin{pmatrix} 1 + u_x^2 & u_x u_y \\ u_x u_y & 1 + u_y^2 \end{pmatrix},$$

its determinant is $g = \det(g_{ij}) = 1 + |\nabla u|^2$, and the inverse

$$(g^{ij}) = \frac{1}{1 + |\nabla u|^2} \begin{pmatrix} 1 + u_y^2 & -u_x u_y \\ -u_x u_y & 1 + u_x^2 \end{pmatrix}.$$

Furthermore, the upwards pointing unit normal (field) is

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|} = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}}.$$

Now we can compute

$$\begin{aligned}
\bar\nabla_{X_1} X_1 &= (0, 0, X_1 u_x) = (0, 0, \nabla u_x \cdot X_1) = (0, 0, u_{xx}), \\
\mathrm{I\!I}(X_1, X_1) &= \big(\bar\nabla_{X_1} X_1\big)^\perp = \frac{(0, 0, u_{xx}) \cdot (-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}} N = \frac{u_{xx}}{\sqrt{1 + |\nabla u|^2}} N; \\
\bar\nabla_{X_1} X_2 &= \bar\nabla_{X_2} X_1 = (0, 0, u_{xy}), \\
\mathrm{I\!I}(X_1, X_2) &= \mathrm{I\!I}(X_2, X_1) = \big(\bar\nabla_{X_1} X_2\big)^\perp = \frac{u_{xy}}{\sqrt{1 + |\nabla u|^2}} N; \\
\bar\nabla_{X_2} X_2 &= (0, 0, u_{yy}), \\
\mathrm{I\!I}(X_2, X_2) &= \big(\bar\nabla_{X_2} X_2\big)^\perp = \frac{u_{yy}}{\sqrt{1 + |\nabla u|^2}} N;
\end{aligned}$$

Hence the mean curvature vector is

$$\begin{aligned}
H &= g^{11}\, \mathrm{I\!I}(X_1, X_1) + 2 g^{12}\, \mathrm{I\!I}(X_1, X_2) + g^{22}\, \mathrm{I\!I}(X_2, X_2) \\
&= \frac{1}{(\sqrt{1 + |\nabla u|^2})^{3/2}} \big((1 + u_y^2) u_{xx} + (1 + u_x^2) u_{yy} - 2 u_x u_y u_{xy}\big) N \\
&= \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) N.
\end{aligned}$$

This is the reason why

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}$$

is also called the *mean curvature operator*.

The Gauss curvature of M is, by the Gauss equation,

$$K = \frac{\langle \mathrm{I\!I}(X_1, X_1), \mathrm{I\!I}(X_2, X_2)\rangle - |\,\mathrm{I\!I}(X_1, X_2)|^2}{|X_1 \wedge X_2|^2} = \frac{\langle u_{xx}N, u_{yy}N\rangle - u_{xy}^2}{(1 + |\nabla u|^2)^2} = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + |\nabla u|^2)^2}.$$

4.3 Bernstein's theorem

We start with introducing the squared norm of the second fundamental form. Let $M^n \subset \mathbb{R}^m$ be a smooth submanifold, $E_1, \dots, E_n$ a local orthonormal frame, and $X_1, \dots, X_n$ an arbitrary frame, with $g_{ij} = \langle X_i, X_j\rangle$. The mean curvature vector is given by

$$H = \sum_{i=1}^{n} \mathrm{I\!I}(E_i, E_i) = \sum_{i,j=1}^{n} g^{ij}\, \mathrm{I\!I}(X_i, X_j).$$

The *squared norm of the second fundamental form* is

$$|\,\mathrm{I\!I}\,|^2 = \sum_{i,j=1}^{n} |\,\mathrm{I\!I}(E_i, E_j)|^2 = \sum_{i,j,k,\ell=1}^{n} g^{ij}g^{k\ell}\langle \mathrm{I\!I}(X_i, X_k), \mathrm{I\!I}(X_j, X_\ell)\rangle.$$

It is often denoted by $|A|^2$. As an example, consider a hypersurface $M \subset \mathbb{R}^m$ and let $\kappa_1, \dots, \kappa_{m-1}$ be the eigenvalues of the Weingarten map with eigenvectors $E_1, \dots, E_{m-1}$ forming a local orthonormal frame. Then

$$|\,\mathrm{I\!I}\,|^2 = \sum_{i,j=1}^{m-1} |\,\mathrm{I\!I}(E_i, E_j)|^2 = \sum_{i=1}^{m-1} \kappa_i^2.$$

By the Cauchy-Schwarz inequality, we get an estimate

$$H^2 = \left(\tfrac{1}{m-1}\operatorname{tr} L\right)^2 \le \frac{1}{m-1}|\,\mathrm{I\!I}\,|^2$$

for the mean curvature H of M.

For the rest of the section we assume that $M \subset \mathbb{R}^3$ is a smooth surface.

Definition 4.4. A smooth surface $M \subset \mathbb{R}^3$ is said to have *finite total curvature* if

$$\int_M |\,\mathrm{I\!I}\,|^2 < \infty.$$

If M is a minimal surface, the Gauss curvature is $K = -\kappa^2$, where $\kappa \ge 0$ and $-\kappa$ are the principal curvatures (eigenvalues of the Weingarten map) and

$$|\,\mathrm{I\!I}\,|^2 = \kappa^2 + (-\kappa)^2 = 2\kappa^2 = -2K.$$

Moreover, $\det dN = -\kappa^2 = K$. Hence a minimal surface $M \subset \mathbb{R}^3$ has finite total curvature if

$$\int_M (-K) = -\int_M K < \infty.$$

In terms of the Gauss map

$$\int_M K = \int_M \det dN$$

which is the (negative) area of the image $N(M)$ with multiplicity counted.

The following lemma provides a tool to estimate the total curvature of a minimal graph. It will be used in the proof of Bernstein's theorem.

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^2$ be an open set, $u\colon \Omega \to \mathbb{R}$ a smooth solution to*

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0,$$

and $M = \{(x, u(x)) \in \mathbb{R}^2 \times \mathbb{R} \colon x \in \Omega\}$ the corresponding minimal graph. Furthermore, let $\eta\colon \Omega \times \mathbb{R} \to [0,\infty)$ be a Lipschitz function such that $\eta|M$ has a compact support. Then there exists an absolute constant C such that

$$\int_M |\,\mathrm{II}\,|^2 \eta^2 \le C \int_M |\nabla^M \eta|^2. \tag{4.6}$$

Proof. Let ω be the area 2-form of $\mathbb{S}^2$. Since ω is closed and the punctured sphere $U = \mathbb{S}^2 \setminus \{0,0,-1)$ is contractible, there exists a 1-form α on U such that $\omega|U = d\alpha$. Let D be the upper hemisphere. Then

$$\sup\{|\alpha(X)| \colon X \in TD,\ |X| = 1\} =: C_\alpha < \infty. \tag{4.7}$$

Let ω_M be the area 2-form of M and let $N\colon M \to \mathbb{S}^2$ be the (upwards pointing) Gauss map, that is,

$$N_p = \frac{(-u_x(x,y), -u_y(x,y), 1)}{\sqrt{1+|\nabla u(x,y)|^2}}$$

at $p = (x, y, u(x,y)) \in M$. Since M is minimal and the pull-back and exterior derivative commute, we have

$$|\,\mathrm{II}\,|^2 \omega_M = -2K\omega_M = -2 \underbrace{\det(dN)\omega_M}_{=N^*\omega} = -2N^*\omega = -2N^*(d\alpha) = -2d(N^*\alpha).$$

On the other hand, by (4.7), we have

$$|N^*\alpha| = \sup_{|X|=1} |N^*\alpha(X)| = \sup_{|X|=1} |\alpha(dN(X))| \le C_\alpha \sqrt{-K} = \frac{C_\alpha |\,\mathrm{II}\,|}{\sqrt{2}}$$

since $N_p \in D$ for all $p \in M$ and $|dN(X)| \le |dN| = \kappa = \sqrt{-K}$ for unit vectors X. Applying Stokes' theorem and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\int_M |\,\mathrm{II}\,|^2 \eta^2 &= \int_M |\,\mathrm{II}\,|^2 \eta^2 \omega_M = -2 \int_M \eta^2 dN^*\alpha \\
&= 4 \int_M \eta d\eta \wedge N^*\alpha - 2 \underbrace{\int_M d(\eta^2 N^*\alpha)}_{=0} \\
&\le 4 \int_M \eta |d\eta| |N^*\alpha| \\
&\le 2\sqrt{2} C_\alpha \int_M \eta |\nabla^M \eta| |\,\mathrm{II}\,| \\
&\le 2\sqrt{2} C_\alpha \left(\int_M |\,\mathrm{II}\,|^2 \eta^2 \right)^{1/2} \left(\int_M |\nabla^M \eta|^2 \right)^{1/2}.
\end{aligned}$$

Hence

$$\int_M |\,\mathrm{II}\,|^2 \eta^2 \le 8C_\alpha^2 \int_M |\nabla^M \eta|^2.$$

□

The idea of the proof of Bernstein's theorem is to show that M is of (conformally) parabolic type. Then it is possible to choose, for every compact set $K \subset M$, a Lipschitz function η such that $\eta = 1$ in K and the right hand side of (4.6) is as small as we wish. The parabolicity follows from the area estimate (1.9). More precisely, we have the following:

Lemma 4.8. *Let Ω, u, and M be as in Lemma 4.5. Suppose that Ω contains a disk $B^2(y, 2^k R)$, $k \in \mathbb{N}$ and denote $M(r) = M \cap B^3((y, u(y)), r)$, $r \le 2^k R$. Then*

$$\int_{M(R)} |\,\mathrm{II}\,|^2 \le C/k.$$

Proof. Without loss of generality, we may assume that $(y, u(y)) = 0 \in \mathbb{R}^3$. We define $\eta\colon \mathbb{R}^3 \to [0,1]$ by setting

$$\eta(x) = \begin{cases} 1, & \text{if } |x| \le R, \\ \frac{\log \frac{2^k R}{|x|}}{k \log 2}, & \text{if } R \le |x| \le 2^k R, \\ 0, & \text{if } |x| \ge 2^k R. \end{cases}$$

Then $\nabla^M \eta(z, u(z)) = \nabla \eta(z, u(z)) = 0$ if $\sqrt{z^2 + u^2(z)} < R$ or $\sqrt{z^2 + u^2(z)} > 2^k R$ and

$$|\nabla^M \eta(z, u(z))| \le |\nabla \eta(z, u(z))| \le \frac{1}{\sqrt{z^2 + u^2(z)} k \log 2} \tag{4.9}$$

for $R \le \sqrt{z^2 + u^2(z)} \le 2^k R$. Combining (4.9) and the area estimate (1.9) we obtain

$$\begin{aligned} \int_{M(R)} |\,\mathrm{II}\,|^2 &\le \int_{M(2^k R)} |\,\mathrm{II}\,|^2 \eta^2 \le c \int_{M(2^k R)} |\nabla^M \eta|^2 \\ &= c \sum_{\ell=1}^{k} \int_{M(2^\ell R) \setminus M(2^{\ell-1} R)} |\nabla^M \eta|^2 \\ &\le c \sum_{\ell=1}^{k} \frac{(2^\ell R)^2}{(2^{\ell-1} R k)^2} \\ &\le \frac{C}{k}. \end{aligned}$$

□

Theorem 4.10 (Bernstein's theorem)**.** *Let $u\colon \mathbb{R}^2 \to \mathbb{R}$ be a solution to*

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0. \tag{4.11}$$

Then u is affine, that is, $u(x, y) = ax + by + c$ for some constants $a, b, c \in \mathbb{R}$. In particular, its graph M is an affine hyperplane.

Proof. Since u is a solution to (4.11) in the whole $\mathbb{R}^2$, we may apply the previous lemma with an arbitrary large R and let $k \to \infty$. Hence $|\,\mathrm{II}\,|^2 \equiv 0$ on M, and therefore all the eigenvalues of the Weingarten map vanish identically on M. Consequently, the Gauss map is a constant and hence M is an affine hyperplane. □

5 Parametric surfaces

5.1 Isothermal coordinates

We want local coordinates for a surface such that the metric is conformal to the Euclidean metric. In dimension 2, it is always possible to find such local (isothermal) coordinates.

Suppose that $\Omega \subset \mathbb{R}^2$ is an open set and $F\colon \Omega \to \mathbb{R}^m,\ F = (F_1, \dots, F_m)$, is an immersion. We denote $M = F\Omega$ and (as usual)

$$X_i = \frac{\partial F}{\partial x_i} = \left(\frac{\partial F_1}{\partial x_i}, \dots, \frac{\partial F_m}{\partial x_i} \right),\ i = 1, 2.$$

Since F is an immersion, every point $z = (x_1, x_2) \in \Omega$ has a neighborhood U such that $F|U$ is an embedding. Then $X_1(z), X_2(z)$ form a basis of $T_{F(z)}U$. We say that F is an *isothermal parametrization* of M if

$$g_{11} = \langle X_1, X_1 \rangle = \lambda^2 = g_{22} = \langle X_2, X_2 \rangle, \quad g_{12} = g_{21} = \langle X_1, X_2 \rangle \equiv 0,$$

where $\lambda = \lambda(x_1, x_2) > 0$. Then $\det(g_{ij}) = \lambda^4$ and $g^{ij} = \lambda^{-2}\delta_{ij}$. The mean curvature vector (of U) at $p = F(z),\ z \in U$, is given by

$$\begin{aligned} H_p &= \left(\sum_{i,j=1}^{2} g^{ij} \frac{\partial^2 F}{\partial x_i \partial x_j}(z) \right)^{\perp} \\ &= \frac{1}{\lambda^2} \left(\frac{\partial^2 F}{\partial x_1^2}(z) + \frac{\partial^2 F}{\partial x_2^2}(z) \right)^{\perp} \\ &= \frac{1}{\lambda^2} \left(\Delta F(z) \right)^{\perp}, \end{aligned}$$

where $\Delta F = (\Delta F_1, \dots, \Delta F_m)$. We prove next that ΔF is, in fact, normal to M. Since

$$\frac{\partial F}{\partial x_1} \cdot \frac{\partial F}{\partial x_1} \equiv \frac{\partial F}{\partial x_2} \cdot \frac{\partial F}{\partial x_2}$$

and

$$\frac{\partial F}{\partial x_2} \cdot \frac{\partial F}{\partial x_1} \equiv 0$$

we get by differentiating the first identify with respect to x_1 and the second with respect to x_2 that

$$\Delta F \cdot \tfrac{\partial F}{\partial x_1} = 0.$$

Similarly, changing the roles of x_1 and x_2 we get

$$\Delta F \cdot \tfrac{\partial F}{\partial x_2} = 0.$$

Hence ΔF is normal to M, and so

$$\Delta F(z) = \lambda^2 H_{F(z)}, \tag{5.2}$$

where $H_{F(z)}$ is the mean curvature vector of FU at $F(z)$, with U a sufficiently small neighborhood of z so that $F|U$ is an embedding. We immediately get the following:

Theorem 5.3. *Let $F\colon \Omega \to \mathbb{R}^m$ be an immersion with isothermal parameters (conformal immersion). Then $M = F\Omega$ is minimal if and only if each coordinate function $F_i\colon \Omega \to \mathbb{R}$ is harmonic, i.e. $\Delta F_i = 0$. Since the target space is $\mathbb{R}^m$, this is equivalent to F being a harmonic mapping.*

Let $F\colon \Omega \to \mathbb{R}^m,\ \Omega \subset \mathbb{R}^2$, be an immersion (not necessarily conformal). We define complex-valued functions $\phi_k\colon \Omega \to \mathbb{C},\ k = 1, \ldots, m$ by setting

$$\phi_k(z) = \frac{\partial F_k}{\partial x_1}(z) - i\frac{\partial F_k}{\partial x_2}(z), \quad z = x_1 + ix_2. \tag{5.4}$$

Then we obtain the identities:

$$\sum_{k=1}^m \phi_k^2 = \sum_{k=1}^m \left(\frac{\partial F_k}{\partial x_1}\right)^2 - \sum_{k=1}^m \left(\frac{\partial F_k}{\partial x_2}\right)^2 - 2i\sum_{k=1}^m \frac{\partial F_k}{\partial x_1} \cdot \frac{\partial F_k}{\partial x_2} = g_{11} - g_{22} - 2ig_{12} \tag{5.5}$$

and

$$\sum_{k=1}^m |\phi_k|^2 = \sum_{k=1}^m \left(\frac{\partial F_k}{\partial x_1}\right)^2 + \sum_{k=1}^m \left(\frac{\partial F_k}{\partial x_2}\right)^2 = g_{11} + g_{22}. \tag{5.6}$$

Hence we have:

Lemma 5.7. *(a) The map $z \mapsto \phi_k(z)$ is analytic if and only if F_k is harmonic.*

(b) Coordinates x_1 and x_2 are isothermal if and only if

$$\sum_{k=1}^m \phi_k^2 \equiv 0. \tag{5.8}$$

(c) If x_1 and x_2 are isothermal, then M is regular if and only if

$$\sum_{k=1}^m |\phi_k|^2 > 0. \tag{5.9}$$

Remark 5.10. Condition (c) requires an explanation: There we have just assumed that $F\colon \Omega \to \mathbb{R}^m$ is differentiable. Furthermore, isothermal coordinates should be interpreted in a broader sense, i.e. we allow $\lambda^2 = g_{11} = g_{22} = 0$, and finally M being regular at $F(z)$ means that the vectors

$$\frac{\partial F}{\partial x_1}(z) \quad \text{and} \quad \frac{\partial F}{\partial x_1}(z)$$

are linearly independent.

Proof. Suppose that F_k is harmonic and write

$$\phi_k = \underbrace{\frac{\partial F_k}{\partial x_1}}_{=u} - i\underbrace{\frac{\partial F_k}{\partial x_2}}_{=v} = u + iv.$$

Then we have

$$\frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_2} = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} = 0.$$

On the other hand,

$$\frac{\partial u}{\partial x_2} = \frac{\partial^2 F}{\partial x_2 \partial x_1} = \frac{\partial^2 F}{\partial x_1 \partial x_2} = -\frac{\partial v}{\partial x_1}.$$

Hence the Cauchy-Riemann equations hold for u and v, and consequently ϕ_k is analytic. Suppose, conversely, that ϕ_k is analytic. Then

$$\begin{aligned} 0 = \frac{\partial}{\partial \bar{z}}\phi_k &= \frac{1}{2}\left(\frac{\partial}{\partial x_1}\phi_k + i\frac{\partial}{\partial x_2}\phi_k\right) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x_1}\left(\frac{\partial F_k}{\partial x_1} - i\frac{\partial F_k}{\partial x_2}\right) + i\frac{\partial}{\partial x_2}\left(\frac{\partial F_k}{\partial x_1} - i\frac{\partial F_k}{\partial x_2}\right)\right) \\ &= \frac{1}{2}\left(\frac{\partial^2 F_k}{\partial x_1^2} - i\frac{\partial^2 F_k}{\partial x_1 \partial x_2} + i\frac{\partial^2 F_k}{\partial x_2 \partial x_1} + \frac{\partial^2 F_k}{\partial x_2^2}\right), \end{aligned}$$

and therefore F_k is harmonic. The condition (b) follows immediately from (5.5) and (c) follows from (5.6). □

Theorem 5.11. *Suppose that $F\colon \Omega \to \mathbb{R}^m$ defines a regular minimal surface M with isothermal parameters. Then the functions ϕ_k in* (5.4) *are analytic satisfying* (5.8) *and* (5.9). *Conversely, let $\phi_1, \phi_2, \ldots, \phi_m$ be analytic functions of $z = x_1 + ix_2$ in a simply connected domain $\Omega \subset \mathbb{C}$ satisfying* (5.8) *and* (5.9). *Then there exists a (regular) minimal surface M parametrized by an immersion $F\colon \Omega \to \mathbb{R}^m$, $F = (F_1, \ldots, F_m)$ such that*

$$\phi_k = \frac{\partial F_k}{\partial x_1} - i\frac{\partial F_k}{\partial x_2}.$$

Proof. Suppose that $F\colon \Omega \to \mathbb{R}^m$ defines a regular minimal surface M with isothermal parameters. By Theorem 5.3, each F_k is harmonic, and hence ϕ_k is analytic satisfying (5.8) and (5.9) by Lemma 5.7. To prove the converse direction, define

$$F_k(\underbrace{x_1, x_2}_{=z}) = \Re \int_{z_0}^{z} \phi_k(\zeta)d\zeta, \tag{5.12}$$

with a fixed $z_0 \in \Omega$. Since ϕ_k is analytic and Ω is simply connected, the map

$$z \mapsto \int_{z_0}^{z} \phi_k(\zeta)d\zeta$$

is a well-defined analytic map in Ω (independent of the choice of a path joining z_0 to z in Ω). Hence

F_k is harmonic. On the other hand,

$$\begin{aligned}
\phi_k(z) &= \frac{\partial}{\partial z}\int_{z_0}^{z}\phi_k(\zeta)d\zeta \\
&= \frac{\partial}{\partial z}\Big(\underbrace{\Re\int_{z_0}^{z}\phi_k(\zeta)d\zeta}_{=F_k} + i\,\underbrace{\Im\int_{z_0}^{z}\phi_k(\zeta)d\zeta}_{=:G_k}\Big) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)(F_k + iG_k) \\
&= \frac{1}{2}\left(\frac{\partial F_k}{\partial x_1} + i\frac{\partial G_k}{\partial x_1} - i\frac{\partial F_k}{\partial x_2} + \frac{\partial G_k}{\partial x_2}\right) \\
&= \frac{1}{2}\left(\frac{\partial F_k}{\partial x_1} - i\frac{\partial F_k}{\partial x_2} - i\frac{\partial F_k}{\partial x_2} + \frac{\partial F_k}{\partial x_1}\right) \\
&= \frac{\partial F_k}{\partial x_1} - i\frac{\partial F_k}{\partial x_2},
\end{aligned}$$

where we used the Cauchy-Riemann equations

$$\frac{\partial F_k}{\partial x_1} = \frac{\partial G_k}{\partial x_2}, \quad \frac{\partial F_k}{\partial x_2} = -\frac{\partial G_k}{\partial x_1}$$

for the real and imaginary part of the analytic function ϕ_k. By Lemma 5.7, parameters x_1 and x_2 are isothermal, and hence M is minimal by Theorem 5.3. $\square$

5.13 Weierstrass-Enneper representation

Consider next the case $m = 3$. As before, let $\phi_k \colon \Omega \to \mathbb{C}$, $k = 1, 2, 3$, be analytic. We can describe explicitly all solutions to

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0. \tag{5.14}$$

Lemma 5.15. *Let $D \subset \mathbb{C}$ be a domain, g an arbitrary meromorphic function in D, and f an analytic function in D such that at each point where g has a pole of order m, f has a zero of order at least $2m$, i.e. fg^2 is analytic. Then the functions*

$$\phi_1 = \tfrac{1}{2}f(1-g^2), \quad \phi_2 = \tfrac{i}{2}f(1+g^2), \quad \phi_3 = fg \tag{5.16}$$

are analytic and satisfy (5.14). *Conversely, every triple (ϕ_1, ϕ_2, ϕ_3) of analytic functions in D satisfying* (5.14) *can be represented in a form* (5.16) *except if $\phi_1 = i\phi_2$ and $\phi_3 = 0$.*

Proof. Functions in (5.16) satisfy

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = \tfrac{1}{4}f^2 - \tfrac{1}{2}f^2g^2 + \tfrac{1}{4}f^2g^4 - \tfrac{1}{4}f^2 - \tfrac{1}{2}f^2g^2 - \tfrac{1}{4}f^2g^4 + f^2g^2 = 0.$$

Conversely, let ϕ_1, ϕ_2, and ϕ_3 be analytic such that (5.14) holds. Suppose $\phi_1 \not\equiv i\phi_2$ and $\phi_3 \not\equiv 0$. Define

$$f = \phi_1 - i\phi_2 \quad \text{and} \quad g = \frac{\phi_3}{\phi_1 - i\phi_2}.$$

Then f is analytic and g is meromorphic. We can write (5.14) as

$$\underbrace{(\phi_1 - i\phi_2)(\phi_1 + i\phi_2)}_{=\phi_1^2+\phi_2^2} = -\phi_3^2.$$

Now, since $\phi_1 \not\equiv i\phi_2$, we have

$$\begin{aligned}\phi_1 + i\phi_2 &= \frac{-\phi_3^2}{\phi_1 - i\phi_2} = \frac{-\phi_3^2}{(\phi_1 - i\phi_2)^2}(\phi_1 - i\phi_2) \\ &= -g^2 f,\end{aligned}$$

and therefore $g^2 f$ is analytic. Moreover,

$$\begin{aligned}\tfrac{1}{2} f(1-g^2) &= \tfrac{1}{2}(\phi_1 - i\phi_2)\left(1 - \frac{\phi_3^2}{(\phi_1 - i\phi_2)^2}\right) \\ &= \tfrac{1}{2}\left(\phi_1 - i\phi_2 - \frac{\phi_3^2}{\phi_1 - i\phi_2}\right) \\ &= \tfrac{1}{2}(\phi_1 - i\phi_2 + \phi_1 + i\phi_2) \\ &= \phi_1,\end{aligned}$$

$$\begin{aligned}\tfrac{i}{2} f(1+g^2) &= \tfrac{i}{2}(\phi_1 - i\phi_2)\left(1 + \frac{\phi_3^2}{(\phi_1 - i\phi_2)^2}\right) \\ &= \tfrac{i}{2}\left(\phi_1 - i\phi_2 + \frac{\phi_3^2}{\phi_1 - i\phi_2}\right) \\ &= \tfrac{i}{2}(\phi_1 - i\phi_2 - \phi_1 - i\phi_2) \\ &= \phi_2,\end{aligned}$$

and

$$fg = \phi_3.$$

Since $-g^2 f = \phi_1 + i\phi_2$ is analytic, the condition on the poles of g and the zeros of f holds. □

Now we are able to prove the following Weierstrass-Enneper representation.

Theorem 5.17. *Let $D \subset \mathbb{C}$ be a simply connected domain, $g\colon D \to \mathbb{C} \cup \{\infty\}$ a meromorphic function, and $f\colon D \to \mathbb{C}$ an analytic function that vanishes only at points ζ where g has a pole and the order of the zero of f at such a point ζ is exactly twice the order of pole of g at ζ. Define $F = (F_1, F_2, F_3)\colon D \to \mathbb{R}^3$ by*

$$F_k(z) = \Re \int_{z_0}^{z} \phi_k(\zeta) d\zeta + c_k, \quad k = 1, 2, 2,$$

where ϕ_1, ϕ_2, and ϕ_3 are as in (5.16)*. Then $F\colon D \to \mathbb{R}^3$ is a minimal immersion.*

Proof. By the previous lemma 5.15, functions ϕ_k are analytic and satisfy (5.14). Moreover,

$$\begin{aligned}|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 &= |\tfrac{1}{2} f(1-g^2)|^2 + |\tfrac{i}{2} f(1+g^2)|^2 + |fg|^2 \\ &= \tfrac{1}{4}|f|^2|1-g^2|^2 + \tfrac{1}{4}|f|^2|1+g^2|^2 + |f|^2|g|^2 \\ &= |f|^2\left(\tfrac{1}{4}(1-g^2)(1-\bar{g}^2) + \tfrac{1}{4}(1+g^2)(1+\bar{g}^2) + |g|^2\right) \\ &= |f|^2\left(\tfrac{1}{4}(1-\bar{g}^2-g^2+|g|^4) + \tfrac{1}{4}(1+\bar{g}^2+g^2+|g|^4) + |g|^2\right) \\ &= |f|^2(\tfrac{1}{2} + |g|^2 + \tfrac{1}{2}|g|^4) \\ &= \tfrac{1}{2}|f|^2(1+|g|^2)^2 > 0\end{aligned}$$

by the assumption on zeros of f and poles on g. Hence by the proof of Theorem 5.11, $F\colon D \to \mathbb{R}^3$ is a minimal immersion. □

We just state the following converse result; see e.g. [O].

Theorem 5.18. *Every simply connected minimal surface in* $\mathbb{R}^3$ *can be represented by* $F = (F_1, F_2, F_3) \colon D \to \mathbb{R}^3$,

$$F_k(z) = \Re \int_{z_0}^{z} \phi_k(\zeta) d\zeta + c_k, \quad k = 1, 2, 3,$$

where the analytic functions ϕ_k *are defined by* (5.16) *with the functions* f *and* g *having the same properties as in Theorem 5.17 and the domain* D *being either the unit disk or the entire complex plane.*

Given a Weierstrass-Enneper data $\{f, g\}$ for a (simply connected) minimal surface M it is possible to express the geometric quantities of M in terms of f and g only. Indeed, let $F \colon D \to \mathbb{R}^3$ be as in the previous theorems and let

$$X_1 = \frac{\partial F}{\partial x_1}, \quad X_2 = \frac{\partial F}{\partial x_2}.$$

Then $g_{ij} = \delta_{ij} \lambda^2$, where

$$\lambda^2 = \left(\frac{|f|(1 + |g|^2)}{2} \right)^2.$$

The Gauss map $N \colon D \to \mathbb{S}^2$ can be written in the form

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|} = \frac{1}{|g|^2 + 1} \big(2\Re(g), 2\Im(g), |g|^2 - 1 \big).$$

Comparing the Gauss map above with the inverse of the stereographic projection from $(0, 0, 1)$, that is, the map $\mathbb{C} \to \mathbb{S}^2$,

$$z = x + iy \mapsto \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right),$$

we obtain the following.

Theorem 5.19. *If* $F \colon D \to \mathbb{R}^3$ *is an isothermal parametrization of a minimal surface, then the corresponding Gauss map* $N \colon D \to \mathbb{S}^2$ *defines a complex analytic function of* D *into the (unit) Riemann sphere.*

Indeed, the Gauss map followed by the stereographic projection gives a meromorphic mapping $D \to \mathbb{S}^2 \to \mathbb{C}$ that is, in fact, the meromorphic function g in the Weierstrass-Enneper representation.

6 The minimal graph equation

6.1 Minimal graphs of codimension 1

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set. Let $u \colon \Omega \to \mathbb{R}$ be a Lipschitz function and consider its graph

$$\Gamma_u = \{(x, u(x)) \colon x \in \Omega\}.$$

We want to justify that also in this higher dimensional ($n \geq 2$) case the (area) volume of Γ_u is

$$\operatorname{Vol}(\Gamma_u) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx.$$

Let $F\colon \Omega \to \mathbb{R}^{n+1}$,

$$F(x) = \big(x_1, x_2, \ldots, x_n, u(x)\big).$$

Since u is Lipschitz, partial derivatives u_{x_i} exist a.e., in fact, u and F are differentiable a.e. by Rademacher's theorem. The matrices of the differential $dF\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ (at a point $x \in \Omega$) and of its adjoint dF^* in standard coordinates $\{e_i\}$ are

$$dF = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ u_{x_1} & u_{x_2} & \cdots & \cdots & u_{x_n} \end{pmatrix} = \begin{pmatrix} I_n \\ \nabla u \end{pmatrix}$$

and

$$dF^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & u_{x_1} \\ 0 & 1 & 0 & \cdots & u_{x_2} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & u_{x_n} \end{pmatrix} = \begin{pmatrix} I_n & \nabla u \end{pmatrix}.$$

Hence

$$(dF^* dF)_{ij} = (\delta_{ij} + u_{x_i} u_{x_j}).$$

Furthermore,

$$\frac{\partial F}{\partial x_i} = e_i + u_{x_i} e_{n+1}$$

and

$$g_{ij} = \frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_j} = \delta_{ij} + u_{x_i} u_{x_j}.$$

Thus

$$(dF^* dF)_{ij} = g_{ij}$$

and

$$\det(dF^* dF)_{ij} = \det g_{ij} = 1 + |\nabla u|^2.$$

The volume form of Γ_u is then

$$\sqrt{1+|\nabla u|^2}\, dx_1 \wedge \cdots \wedge dx_n,$$

and therefore the volume of Γ_u is

$$\mathrm{Vol}(\Gamma_u) = \int_\Omega \sqrt{1+|\nabla u|^2}\, dx.$$

As in the case $n = 2$ a function $u \in C^2(\Omega)$ is a critical point of the volume functional

$$\mathcal{F}(u) := \int_\Omega \sqrt{1+|\nabla u|^2}\, dx,$$

that is

$$\frac{d}{dt}\mathcal{F}(u + t\varphi)_{|t=0} = 0, \quad \varphi \in C_0^\infty(\Omega),$$

if

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0. \tag{6.2}$$

Also now, a critical point u of $\mathcal{F}$ minimizes the volume of graphs Γ_v among all functions $v\colon \Omega \to \mathbb{R}$, with $v = u$ on $\partial\Omega$. Here, of course, we assume that functions u and v are continuous in $\bar{\Omega}$. We want to prove (partially) the following theorem.

Theorem 6.3 (Jenkins-Serrin)**.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain whose boundary has nonnegative scalar mean curvature (with respect to inwards pointing normal). Then for each $\psi \in C^{2,\alpha}(\bar{\Omega})$ there exists a unique $u \in C^\infty(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$ that solves the minimal graph equation* (6.2) *in Ω with boundary values $u|\partial\Omega = \psi|\partial\Omega$.*

Recall that $C^k(\bar{\Omega})$ denotes the class of all functions whose all derivatives of order $\leq k$ have continuous extensions to $\bar{\Omega}$ and $C^{k,\alpha}(\bar{\Omega})$ is the subset of those functions in $C^k(\bar{\Omega})$ whose k^{th}-order partial derivatives are globally Hölder-continuous with exponent α.

Our strategy will be the following:

1. First we will prove the existence of minimizers of the volume functional within *subclasses* of Lipschitz functions with uniformly bounded Lipschitz constants. Here we employ so-called direct methods in calculus of variations. In fact, we will consider a quite general class of variational integrals.

2. As a second step we consider so-called *a priori* gradient estimates.

3. In the third step we reduce (interior) gradient estimates to boundary gradient estimates. Here we will focus on the volume functional and will obtain boundary gradient estimates by constructing suitable "barriers". It is in this step where the assumption $\partial\Omega$ having non-negative mean curvature (w.r.t. inwards pointing normal) is needed.

6.4 Direct methods

We consider variational integrals

$$\mathcal{F}(u) := \int_\Omega F(\nabla u)dx,$$

where functions u belong to some subspace $\mathcal{A}$ of Lipschitz functions in a bounded domain $\Omega \subset \mathbb{R}^n$. The class $\mathcal{A}$ is equipped with a topology that makes $\mathcal{F}$ *lower semicontinuous* (see (6.5) below) and minimizing sequences (sequentially) compact in $\mathcal{A}$. More precisely, if (u_i) is a minimizing sequence, i.e. $u_i \in \mathcal{A}$ and

$$\lim_{i\to\infty} \mathcal{F}(u_i) = \inf_{u\in\mathcal{A}} \mathcal{F}(u),$$

then there exists a converging subsequence $u_{i_j} \to \bar{u} \in \mathcal{A}$ and

$$\mathcal{F}(\bar{u}) \leq \liminf_{j\to\infty} \mathcal{F}(u_{i_j}). \tag{6.5}$$

It then follows that $\bar{u}$ is a minimizer of $\mathcal{F}$ among functions in $\mathcal{A}$.

Here are some examples of variational integrals:

$$
\begin{aligned}
\mathcal{F}(u) &= \int_\Omega |\nabla u|^2 dx, \\
\mathcal{F}(u) &= \int_\Omega |\nabla u|^p dx, \ 1 < p < \infty, \\
\mathcal{F}(u) &= \int_\Omega \sqrt{1 + |\nabla u|^2} dx, \\
\mathcal{F}(u) &= \int_\Omega \left(1 + |\nabla u|^k\right)^{1/k} dx, \ k > 0, \\
\mathcal{F}(u) &= \int_\Omega \exp |\nabla u|^2, \\
\mathcal{F}(u) &= \int_\Omega |\nabla u|^2 \log(1 + \nabla |u|^2), \\
\mathcal{F}(u) &= \int_\Omega \left(\sum_{i=1}^{n-1} |u_{x_i}|^2 + |u_{x_n}|^k \right), \ k > 0.
\end{aligned}
$$

All of these variational integrals are defined among Lipschitz functions, although some other (larger) function spaces, like Sobolev spaces, would be more natural to work with. Furthermore, in all of these cases the *variational kernel* $F\colon \mathbb{R}^n \to [0, \infty)$ is convex. The convexity of F is very useful in obtaining the lower semicontinuity (6.5) of $\mathcal{F}$.

Suppose that $F\colon \mathbb{R}^n \to [0, \infty)$ is convex and C^1. Then

$$F(x) - F(y) \geq \nabla F(y) \cdot (x - y) \tag{6.6}$$

for all $x, y \in \mathbb{R}^n$. Let us study the (possible) lower semicontinuity of $\mathcal{F}$ among Lipschitz functions

$$\operatorname{Lip}(\Omega) = \{u\colon \Omega \to \mathbb{R} \mid u \text{ Lipschitz}\}.$$

We define a semi-norm

$$|u|_\Omega := \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|} : x, y \in \Omega, \ x \neq y \right\}.$$

Let (u_j) be a sequence in $\operatorname{Lip}(\Omega)$. By (6.6), we have

$$\mathcal{F}(u) = \int_\Omega F(\nabla u) \leq \int_\Omega F(\nabla u_j) - \int_\Omega \nabla F(\nabla u) \cdot (\nabla u_j - \nabla u) \tag{6.7}$$

for all $u \in \operatorname{Lip}(\Omega)$. Furthermore, since ∇F is continuous, we have $|\nabla F(\nabla u)| \in L^\infty(\Omega)$. Suppose that $\nabla u_j \to \nabla u$ weakly in $L^1(\Omega)$. Then

$$\int_\Omega \nabla F(\nabla u) \cdot \nabla u_j \to \int_\Omega \nabla F(\nabla u) \cdot \nabla u,$$

and therefore

$$\int_\Omega \nabla F(\nabla u) \cdot (\nabla u_j - \nabla u) \to 0$$

and finally

$$\mathcal{F}(u) \leq \liminf_{j \to \infty} \mathcal{F}(u_j)$$

by (6.7).

Let

$$\mathrm{Lip}_k(\Omega) = \{u \in \mathrm{Lip}(\Omega) \colon |u|_\Omega \le k\}.$$

Lemma 6.8. *Suppose that $F \colon \mathbb{R}^n \to \mathbb{R}$ is convex and C^1-smooth. Then the variational integral $\mathcal{F} \colon \mathrm{Lip}(\Omega) \to \mathbb{R}$,*

$$\mathcal{F}(u) = \int_\Omega F(\nabla u) dx,$$

is lower semicontinuous with respect to the uniform convergence of sequences with equibounded Lipschitz seminorm and with fixed boundary values on $\partial\Omega$. More precisely, if $u_j \in \mathrm{Lip}_k(\bar{\Omega})$, $u_j|\partial\Omega = \varphi$, and $u_j \to u$ uniformly in Ω, then

$$\mathcal{F}(u) \le \liminf_{j\to\infty} \mathcal{F}(u_j).$$

Proof. First we observe that $u \in \mathrm{Lip}_k(\Omega)$. Approximate the mapping $x \mapsto \nabla F(\nabla u)$ in $L^1(\Omega)$ by smooth mappings

$$F_\varepsilon = (F_{1,\varepsilon}, F_{2,\varepsilon}, \dots, F_{n,\varepsilon}) \colon \Omega \to \mathbb{R}^n.$$

Now

$$\int_\Omega \nabla F(\nabla u) \cdot (\nabla u_j - \nabla u) = \int_\Omega (\nabla F(\nabla u) - F_\varepsilon) \cdot (\nabla u_j - \nabla u) + \int_\Omega F_\varepsilon \cdot (\nabla u_j - \nabla u).$$

Since $F_\varepsilon \to \nabla F(\nabla u)$ in $L^1(\Omega)$ as $\varepsilon \to 0$ and $u, u_j \in \mathrm{Lip}_k(\Omega)$, we have for all $\delta > 0$ that

$$\Big|\int_\Omega (\nabla F(\nabla u) - F_\varepsilon) \cdot (\nabla u_j - \nabla u)\Big| \le \int_\Omega |\nabla F(\nabla u) - F_\varepsilon||\nabla u_j - \nabla u| \le 2k \int_\Omega |\nabla F(\nabla u) - F_\varepsilon| \le \delta$$

for all $\varepsilon > 0$ small enough. On the other hand, by integration by parts

$$\int_\Omega F_\varepsilon \cdot (\nabla u_j - \nabla u) = \underbrace{\int_\Omega \mathrm{div}\,((u_j - u)F_\varepsilon)}_{=0} - \int_\Omega (\mathrm{div}\, F_\varepsilon)(u_j - u) = -\int_\Omega (\mathrm{div}\, F_\varepsilon)(u_j - u) \to 0$$

since $u_j \to u$ uniformly. Note that the Lipschitz vector field $(u_j - u)F_\varepsilon$ vanishes on $\partial\Omega$. By the convexity of F and the reasoning above, we have for all $\delta > 0$ that

$$\begin{aligned}
\mathcal{F}(u) = \int_\Omega F(\nabla u) &\le \int_\Omega F(\nabla u_j) - \int_\Omega \nabla F(\nabla u) \cdot (\nabla u_j - \nabla u) \\
&= \mathcal{F}(u_j) - \int_\Omega (\nabla F(\nabla u) - F_\varepsilon) \cdot (\nabla u_j - \nabla u) - \int_\Omega F_\varepsilon \cdot (\nabla u_j - \nabla u) \\
&= \mathcal{F}(u_j) - \underbrace{\int_\Omega (\nabla F(\nabla u) - F_\varepsilon) \cdot (\nabla u_j - \nabla u)}_{\le \delta} + \int_\Omega (\mathrm{div}\, F_\varepsilon)(u_j - u) \\
&\le \mathcal{F}(u_j) + \int_\Omega (\mathrm{div}\, F_\varepsilon)(u_j - u) + \delta
\end{aligned}$$

as soon as $\varepsilon > 0$ is small enough. It follows that $\mathcal{F}(u) \le \liminf_{j\to\infty} \mathcal{F}(u_j) + \delta$ for all $\delta > 0$, and therefore

$$\mathcal{F}(u) \le \liminf_{j\to\infty} \mathcal{F}(u_j).$$

□

Theorem 6.9. *Suppose that $F\colon \mathbb{R}^n \to \mathbb{R}$ is convex and C^1-smooth and let $\varphi \in \mathrm{Lip}_k(\partial\Omega)$. Then the variational integral*

$$\mathcal{F}(u) = \int_\Omega F(\nabla u)dx$$

has a minimizer in the class $\mathcal{A}_k = \{u \in \mathrm{Lip}_k(\bar\Omega)\colon u = \varphi \text{ on } \partial\Omega\}$.

Proof. By the McShane-Whitney extension theorem, φ can be extended to a function $\varphi \in \mathrm{Lip}_k(\bar\Omega)$. In particular, $\mathcal{A}_k$ is non-empty. Let (u_j) be a minimizing sequence in $\mathcal{A}_k$. Then it is equibounded and equicontinuous, hence by the Ascoli-Arzelá theorem, there is a subsequence (still denoted by (u_j) and $\bar u \in \mathcal{A}_k$ such that $u_j \to \bar u$ uniformly. By Lemma 6.8,

$$\mathcal{F}(\bar u) \le \liminf_{j\to\infty} \mathcal{F}(u_j) = \inf\{\mathcal{F}(u)\colon u \in \mathcal{A}_k\}.$$

Since $\bar u \in \mathcal{A}_k$, it is a minimizer in $\mathcal{A}_k$. □

Remark 6.10. The function $\bar u$ found above need not be a minimizer of $\mathcal{F}(u)$ among all Lipschitz functions u in $\bar\Omega$ with $u = \varphi$ on $\partial\Omega$.

However, we have the following simple but useful result.

Lemma 6.11. *Suppose that a minimizer $\bar u$ of $\mathcal{F}(u)$ in $\mathcal{A}_k$ satisfies $|\bar u|_\Omega < k$. Then $\bar u$ is a minimizer of $\mathcal{F}(u)$ in $\mathcal{A} = \{u \in \mathrm{Lip}(\bar\Omega)\colon u = \varphi \text{ on } \partial\Omega\}$.*

Proof. Let $w \in \mathrm{Lip}(\bar\Omega)$, with $w = \varphi$ on $\partial\Omega$. Since $|\bar u|_\Omega < k$, there exists $t \in (0,1)$ such that

$$|tw + (1-t)\bar u|_\Omega \le t\underbrace{|w|_\Omega}_{<\infty} + (1-t)\underbrace{|\bar u|_\Omega}_{<k} \le k,$$

and therefore $tw + (1-t)\bar u \in \mathrm{Lip}_k(\bar\Omega)$. Since $\bar u$ is a minimizer of $\mathcal{F}(u)$ in $\mathcal{A}_k$ and F is convex, we have

$$\mathcal{F}(\bar u) \le \mathcal{F}(tw + (1-t)\bar u) \le t\mathcal{F}(w) + (1-t)\mathcal{F}(\bar u),$$

and so

$$\mathcal{F}(\bar u) \le \mathcal{F}(w).$$

□

6.12 Gradient estimates

In this subsection we assume that all Lipschitz functions in Ω are continuously extended to $\bar\Omega$.

Definition 6.13. Given a variational integral

$$\mathcal{F}(u) = \int_\Omega F(\nabla u)dx,$$

we say that a function u is a *superminimizer* in $\mathrm{Lip}(\Omega)$ (resp. in $\mathrm{Lip}_k(\Omega)$) if

$$\mathcal{F}(u+\varphi) \ge \mathcal{F}(u)$$

for all $\varphi \in \mathrm{Lip}(\bar\Omega)$ (resp. $\mathrm{Lip}_k(\bar\Omega)$), $\varphi \ge 0$, and $\varphi = 0$ on $\partial\Omega$. Furthermore, a function v is a *subminimizer* in $\mathrm{Lip}(\Omega)$ (resp. in $\mathrm{Lip}_k(\bar\Omega)$) if

$$\mathcal{F}(v-\varphi) \ge \mathcal{F}(v)$$

for all $\varphi \in \mathrm{Lip}(\bar\Omega)$ (resp. $\mathrm{Lip}_k(\bar\Omega)$), $\varphi \ge 0$, and $\varphi = 0$ on $\partial\Omega$.

It is easily seen that a function is a minimizer if and only if it is both super- and subminimizer.

Theorem 6.14 (Comparison principle)**.** *Suppose that F is strictly convex and that u is a superminimizer in* $\mathrm{Lip}(\bar{\Omega})$ *(resp.* $\mathrm{Lip}_k(\bar{\Omega})$*) and v is a subminimizer in* $\mathrm{Lip}(\bar{\Omega})$ *(resp.* $\mathrm{Lip}_k(\bar{\Omega})$*) such that $u \geq v$ on $\partial\Omega$. Then $u \geq v$ in Ω.*

Proof. Suppose on the contrary that the open set

$$K = \{x \in \Omega \colon v(x) > u(x)\}$$

is non-empty. Define functions $\tilde{u}$ and $\tilde{v}$ by

$$\tilde{u}(x) = \begin{cases} u(x), & \text{if } x \in \bar{\Omega} \setminus K; \\ v(x), & \text{if } x \in K, \end{cases} \qquad \tilde{v}(x) = \begin{cases} v(x), & \text{if } x \in \bar{\Omega} \setminus K; \\ u(x), & \text{if } x \in K. \end{cases}$$

Then $u \leq \tilde{u},\ v \geq \tilde{v}$ in Ω and $u = \tilde{u},\ v = \tilde{v}$ on $\partial\Omega$. Hence $\mathcal{F}(u) \leq \mathcal{F}(\tilde{u})$ and $\mathcal{F}(v) \leq \mathcal{F}(\tilde{v})$, and therefore

$$\begin{aligned} \mathcal{F}(u) &= \int_{\Omega\setminus K} F(\nabla u) + \int_K F(\nabla u) = \int_{\Omega\setminus K} F(\nabla \tilde{u}) + \int_K F(\nabla u) \\ &\leq \mathcal{F}(\tilde{u}) = \int_{\Omega\setminus K} F(\nabla \tilde{u}) + \int_K F(\nabla \tilde{u}) \\ &= \int_{\Omega\setminus K} F(\nabla \tilde{u}) + \int_K F(\nabla v). \end{aligned}$$

Hence

$$\int_K F(\nabla u) \leq \int_K F(\nabla v).$$

Similarly,

$$\int_K F(\nabla v) \leq \int_K F(\nabla u),$$

and so

$$\int_K F(\nabla u) = \int_K F(\nabla v).$$

Since $m(K) > 0$, the strict convexity of F implies that

$$\int_K F\left(\tfrac{\nabla u + \nabla v}{2}\right) < \frac{1}{2}\int_K F(\nabla u) + \frac{1}{2}\int_K F(\nabla v) = \int_K F(\nabla v).$$

This leads to a contradiction with the assumption v being a subminimizer in $\mathrm{Lip}(\Omega)$ (resp. in $\mathrm{Lip}_k(\Omega)$) since the function w,

$$w(x) = \begin{cases} v(x), & \text{if } x \in \bar{\Omega} \setminus K; \\ \frac{v(x)+u(x)}{2}, & \text{if } x \in K, \end{cases}$$

belongs to $\mathrm{Lip}(\bar{\Omega})$ (resp. $\mathrm{Lip}_k(\bar{\Omega})$), $w \leq v$, and $w = v$ on $\partial\Omega$, but

$$\mathcal{F}(w) = \int_{\Omega\setminus K} F(\nabla v) + \int_K F\left(\tfrac{\nabla u + \nabla v}{2}\right) < \int_{\Omega\setminus K} F(\nabla v) + \int_K F(\nabla v) = \mathcal{F}(v).$$

□

From now on we assume that $F\colon \mathbb{R}^n \to [0,\infty)$ is strictly convex and C^1.

Theorem 6.15 (Maximum principle). *Suppose that u is a superminimizer in $\mathrm{Lip}(\bar\Omega)$ (resp. in $\mathrm{Lip}_k(\bar\Omega)$) and v is a subminimizer in $\mathrm{Lip}(\bar\Omega)$ (resp. in $\mathrm{Lip}_k(\bar\Omega)$). Then*

$$\sup_\Omega (v-u) = \sup_{\partial\Omega}(v-u).$$

In particular, if u and v are minimizers (in $\mathrm{Lip}(\Omega)$ or $\mathrm{Lip}_k(\Omega)$), then

$$\sup_\Omega |v-u| = \sup_{\partial\Omega}|v-u|.$$

Proof. Since

$$u + \sup_{\partial\Omega}(v-u)$$

is a superminimizer in $\mathrm{Lip}(\Omega)$ (resp. in $\mathrm{Lip}_k(\Omega)$) and

$$u + \sup_{\partial\Omega}(v-u) \ge v$$

on $\partial\Omega$, it follows from the comparison principle that

$$v \le u + \sup_{\partial\Omega}(v-u)$$

in Ω. Hence

$$\sup_\Omega (v-u) \le \sup_{\partial\Omega}(v-u).$$

The equality follows from the continuity (in $\bar\Omega$) of u and v. □

Lemma 6.16 (Haar-Radò). *Let u be a minimizer (in $\mathrm{Lip}(\Omega)$ or in $\mathrm{Lip}_k(\Omega)$). Then*

$$\sup\left\{\frac{|u(x)-u(y)|}{|x-y|}\colon x,y\in\Omega,\ x\ne y\right\} = \sup\left\{\frac{|u(x)-u(y)|}{|x-y|}\colon x\in\Omega,\ y\in\partial\Omega\right\}.$$

Proof. Let $x_1,x_2\in\Omega,\ x_1\ne x_2$, and let $\tau = x_2-x_1$. Define a set Ω_τ and a function $u_\tau\colon\Omega_\tau\to\mathbb{R}$ by

$$\Omega_\tau = \{x-\tau\colon x\in\Omega\} \text{ and } u_\tau(x) = u(x+\tau).$$

Then $\Omega_\tau\cap\Omega$ contains x_1 and thus it is a non-empty open set. Both u and u_τ are minimizers in $\mathrm{Lip}(\Omega_\tau\cap\Omega)$ (or in $\mathrm{Lip}_k(\Omega_\tau\cap\Omega)$). By the Maximum principle,

$$\begin{aligned}|u(x_1)-u(x_2)| &= |u(x_1)-u_\tau(x_1)| \le \sup_{\Omega_\tau\cap\Omega}|u-u_\tau| \le \sup_{\partial(\Omega_\tau\cap\Omega)}|u-u_\tau| \\ &= \max_{\partial(\Omega_\tau\cap\Omega)}|u-u_\tau| = |u(z)-u_\tau(z)| = |u(z)-u(z+\tau)|\end{aligned}$$

for some $z\in\partial(\Omega_\tau\cap\Omega)$. Next we observe that $\partial(\Omega_\tau\cap\Omega)\subset\partial\Omega_\tau\cup\partial\Omega$ which implies that at least one of the points z or $z+\tau$ belongs to $\partial\Omega$. Furthermore, both z and $z+\tau$ belong to $\bar\Omega$ since $z\in\partial(\Omega_\tau\cap\Omega)$. Hence

$$\frac{|u(x_1)-u(x_2)|}{|x_1-x_2|} = \frac{|u(x_1)-u(x_2)|}{|\tau|} \le \frac{|u(z)-u(z+\tau)|}{|\tau|} \le \sup\left\{\frac{|u(x)-u(y)|}{|x-y|}\colon x\in\Omega,\ y\in\partial\Omega\right\},$$

and so

$$\sup\left\{\frac{|u(x)-u(y)|}{|x-y|}\colon x,y\in\Omega,\ x\ne y\right\} \le \sup\left\{\frac{|u(x)-u(y)|}{|x-y|}\colon x\in\Omega,\ y\in\partial\Omega\right\}.$$

The equality follows from the continuity of u in $\bar\Omega$. □

6.17 Barriers

For $x \in \bar{\Omega}$ and $t > 0$, we denote

$$d(x) = \operatorname{dist}(x, \partial\Omega) = \min\{|x-y| \colon y \in \partial\Omega\},$$
$$\Sigma_t = \{y \in \Omega \colon d(y) < t\},$$

and

$$\Gamma_t = \{y \in \Omega \colon d(y) = t\}.$$

Definition 6.18. Let $\varphi \in \operatorname{Lip}(\partial\Omega)$. An *upper barrier relative to* φ is a function $v^+ \in \operatorname{Lip}(\bar{\Sigma}_{t_0})$, for some $t_0 > 0$, satisfying

(i) $v^+ = \varphi$ on $\partial\Omega$ and $v^+ \geq \sup_{\partial\Omega} \varphi$ on Γ_{t_0}, and

(ii) v^+ is a superminimizer in $\operatorname{Lip}(\Sigma_{t_0})$.

Similarly, a *lower barrier relative to* φ is a function $v^- \in \operatorname{Lip}(\bar{\Sigma}_{t_0})$ such that

(i) $v^- = \varphi$ on $\partial\Omega$ and $v^- \leq \inf_{\partial\Omega} \varphi$ on Γ_{t_0}, and

(ii) v^- is a subminimizer in $\operatorname{Lip}(\Sigma_{t_0})$.

Theorem 6.19. *Let $\varphi \in \operatorname{Lip}(\partial\Omega)$ and suppose that there exist an upper barrier v^+ and a lower barrier v^- relative to φ. Then the variational integral*

$$\mathcal{F}(u) = \int_\Omega F(\nabla u)dx$$

has a minimizer in $\mathcal{A} = \{u \in \operatorname{Lip}(\bar{\Omega}) \colon u = \varphi \text{ on } \partial\Omega\}$.

Proof. Let $Q = \max\{|v^+|_{\Sigma_{t_0}}, |v^-|_{\Sigma_{t_0}}\}$ and fix $k > Q$. By Theorem 6.9, there exists a minimizer u in the class $\mathcal{A}_k = \{v \in \operatorname{Lip}_k(\bar{\Omega}) \colon v = \varphi \text{ on } \partial\Omega\}$. Then $u|\Sigma_{t_0}$ is a minimizer of the variational integral

$$\int_{\Sigma_{t_0}} F(\nabla u)dx$$

among functions in $\tilde{\mathcal{A}}_k = \{v \in \operatorname{Lip}_k(\bar{\Sigma}_{t_0}) \colon v = u \text{ on } \partial\Sigma_{t_0}\}$. By the Maximum principle (with u and 0),

$$\sup_\Omega |u| = \sup_{\partial\Omega} |u| = \sup_{\partial\Omega} |\varphi|,$$
$$\sup_\Omega |-u| = \sup_{\partial\Omega} |-u| = \sup_{\partial\Omega} |-\varphi|,$$

and therefore

$$\inf_{\partial\Omega} \varphi \leq u(x) \leq \sup_{\partial\Omega} \quad \forall x \in \Omega.$$

In particular,

$$v^-(x) \leq \inf_{\partial\Omega} \varphi \leq u(x) \leq \sup_{\partial\Omega} \leq v^+(x) \quad \forall x \in \Gamma_{t_0},$$

and hence by the Comparison principle

$$v^-(x) \leq u(x) \leq v^+(x) \quad \forall x \in \Sigma_{t_0}.$$

Note that $v^- = v^+ = u = \varphi$ on $\partial\Omega$. We obtain

$$|u(x) - u(y)| \leq Q|x - y|$$

for every $x \in \Sigma_{t_0}$ and $y \in \partial\Omega$. Indeed, if $u(x) \geq u(y)$, we have

$$|u(x) - u(y)| = u(x) - u(y) = u(x) - v^+(y) \leq v^+(x) - v^+(y) \leq Q|x - y|,$$

and if $u(x) \leq u(y)$ we have

$$|u(x) - u(y)| = u(y) - u(x) = v^-(y) - u(x) \leq v^-(y) - v^-(x) \leq Q|x - y|.$$

Suppose then that $x \in \Omega$, with $d(x) \geq t_0$, and that $y \in \partial\Omega$. Since

$$\inf_{\partial\Omega} \varphi \leq u \leq \sup_{\partial\Omega}$$

in Ω, we have

$$|u(x) - u(y)| \leq \max\{\sup_{\partial\Omega} \varphi - u(y), u(y) - \inf_{\partial\Omega} \varphi\}.$$

Let $z \in \Gamma_{t_0}$ be a point such that

$$|x - y| = |x - z| + |z - y|.$$

Then

$$\sup_{\partial\Omega} \varphi - u(y) \leq v^+(z) - u(y) = v^+(z) - v^+(y) \leq Q|z - y| \leq Q|x - y|$$

and

$$u(y) - \inf_{\partial\Omega} \varphi \leq u(y) - v^-(z) = v^-(y) - v^-(z) \leq Q|z - y| \leq Q|x - y|.$$

Hence

$$|u(x) - u(y)| \leq Q|x - y|$$

for all $x \in \Omega$ and $y \in \partial\Omega$. By Haar-Radò lemma 6.16,

$$|u|_\Omega \leq Q < k,$$

and therefore u is a minimizer in $\mathcal{A} = \{v \in \mathrm{Lip}(\bar{\Omega}) \colon v = \varphi \text{ on } \partial\Omega\}$. $\square$

6.20 Construction of barriers for the minimal graph equation

In this subsection we consider the volume functional

$$\mathcal{V}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set.

Suppose that $u \in \mathrm{Lip}(\Omega)$ is a superminimizer of $\mathcal{V}$ and let $\varphi \in \mathrm{Lip}(\bar{\Omega})$, with $\varphi \geq 0$ and $\varphi = 0$ on $\partial\Omega$. Since

$$\mathcal{V}(u) \leq \mathcal{V}(u + t\varphi) \quad \forall t \geq 0,$$

we have

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(u+t\varphi)_{|t=0+} &= \lim_{t\to 0+}\frac{\mathcal{V}(u+t\varphi)-\mathcal{V}(u)}{t}\\ &= \lim_{t\to 0+}\frac{1}{t}\int_\Omega\left(\sqrt{1+|\nabla u+t\nabla\varphi|^2}-\sqrt{1+|\nabla u|^2}\right)\\ &= \int_\Omega\lim_{t\to 0+}\frac{\sqrt{1+|\nabla u+t\nabla\varphi|^2}-\sqrt{1+|\nabla u|^2}}{t}\\ &= \int_\Omega\frac{\nabla u\cdot\nabla\varphi}{\sqrt{1+|\nabla u|^2}}\geq 0.\end{aligned}$$

Furthermore, supposing that $u\in C^2(\Omega)$ and using integration by parts, we see that the above is equivalent to

$$-\int_\Omega\varphi\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\geq 0$$

for all $\varphi\in\mathrm{Lip}(\bar\Omega)$, with $\varphi\geq 0$ and $\varphi=0$ on $\partial\Omega$. Hence, if $u\in C^2(\Omega)$ is a superminimizer, then

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\leq 0. \tag{6.21}$$

Similarly, if $v\in\mathrm{Lip}(\Omega)$ is a subminimizer of $\mathcal{V}$, then

$$\int_\Omega\frac{\nabla v\cdot\nabla\varphi}{\sqrt{1+|\nabla v|^2}}\leq 0$$

for all $\varphi\in\mathrm{Lip}(\bar\Omega)$, with $\varphi\geq 0$ and $\varphi=0$ on $\partial\Omega$. If, in addition, $v\in C^2(\Omega)$, we obtain

$$\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)\geq 0. \tag{6.22}$$

These inequalities are equivalent to

$$\mathcal{E}(u):=(1+|\nabla u|^2)\Delta u-\sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_ix_j}\leq 0 \tag{6.23}$$

and

$$\mathcal{E}(v)=(1+|\nabla v|^2)\Delta v-\sum_{i,j=1}^n v_{x_i}v_{x_j}v_{x_ix_j}\geq 0. \tag{6.24}$$

Suppose that $\Omega\subset\mathbb{R}^n$ is a bounded open set, with a C^2-smooth boundary $\partial\Omega$. In the next subsection we will prove that:

(i) the distance function $d\colon\bar\Omega\to[0,\infty)$, $d(x)=\mathrm{dist}(x,\partial\Omega)$, is C^2 in some $\bar\Sigma_{t_0}$;

(ii) furthermore, if $x\in\Gamma_{t_0}$, then $-\Delta d(x)$ is the sum of the principal curvatures of Γ_{t_0} with respect to the inwards pointing unit normal; and that

(iii) $\Delta d(x)$ decreases when $x\in\bar\Sigma_{t_0}$ moves towards the interior of Ω along a normal to $\partial\Omega$.

Hence assuming that $\partial\Omega$ has non-negative mean curvature with respect to inwards pointing normal, we obtain

$$\Delta d(x) \le 0 \tag{6.25}$$

in $\bar{\Sigma}_{t_0}$.

Suppose that $\partial\Omega$ has non-negative mean curvature with respect to inwards pointing normal and that the boundary value function φ belongs to $C^2(\bar{\Omega})$. Under these assumptions we will construct an upper barrier v relative to $\varphi|\partial\Omega$ of the form

$$v(x) = \varphi(x) + \psi\big(d(x)\big),$$

where $\psi\colon [0,R] \to [0,\infty)$ is a C^2-function such that

$$\psi(0) = 0, \quad \psi'(t) \ge 1, \quad \psi''(t) < 0,$$

and

$$\psi(R) \ge L := 2\sup_{\Omega} |\varphi|$$

for some $R < t_0$ determined later. Note that the assumption $\psi(R) \ge 2\sup_{\Omega}|\varphi|$ implies that

$$v(x) \ge \sup_{\partial\Omega} \varphi \quad \forall x \in \Gamma_R.$$

Since $v = \varphi$ on $\partial\Omega$ by the assumption $\psi(0) = 0$, the condition (i) in Definition 6.18 holds. Therefore, to show that v is an upper barrier relative to φ, we need to choose ψ so that v will be a superminimizer in Σ_R, i.e.

$$\mathcal{E}(v) = (1 + |\nabla v|^2)\Delta v - \sum_{i,j=1}^{n} v_{x_i} v_{x_j} v_{x_i x_j} \le 0.$$

To simplify the notation we abbreviate

$$v_i = v_{x_i}, \; v_{ij} = v_{x_i x_j}, \; \psi' = (\psi') \circ d \;\; \text{etc.}$$

We have

$$\begin{aligned} v_i &= \varphi_i + \psi' d_i, \\ v_{ij} &= \varphi_{ij} + \psi d_{ij} + \psi'' d_i d_j, \\ \nabla v &= \nabla\varphi + \psi' \nabla d, \\ \Delta v &= \Delta\varphi + \psi' \Delta d + \psi'', \end{aligned}$$

and

$$1 + |\nabla v|^2 = 1 + |\nabla\varphi|^2 + (\psi')^2 + 2\psi' \sum_i \varphi_i d_i,$$

where we used the fact that $|\nabla d| \equiv 1$. We also have

$$0 \equiv \tfrac{1}{2}\big(|\nabla d|^2\big)_j = \sum_{i=1}^{n} d_i d_{ij} \quad \forall j.$$

We compute

$$
\begin{aligned}
\mathcal{E}(v) &= (1+|\nabla v|^2)\Delta v - \sum_{ij} v_i v_j v_{ij} \\
&= \big(1+|\nabla\varphi|^2 + (\psi')^2 + 2\psi' \sum_i \varphi_i d_i\big)\big(\Delta\varphi + \psi'\Delta d + \psi''\big) \\
&- \sum_{ij} \big(\varphi_i + \psi' d_i\big)\big(\varphi_j + \psi' d_j\big)\big(\varphi_{ij} + \psi' d_{ij} + \psi'' d_i d_j\big) \\
&= (1+|\nabla\varphi|^2)\Delta\varphi - \sum_{ij} \varphi_i\varphi_j\varphi_{ij} \\
&+ \psi'\big[2\sum_i \varphi_i d_i \Delta\varphi + (1+|\nabla\varphi|^2)\Delta d - 2\sum_{ij} d_i\varphi_j\varphi_{ij} - \sum_{ij}\varphi_i\varphi_j d_{ij}\big] \\
&+ (\psi')^2\big[\Delta\varphi + 2\sum_i \varphi_i d_i \Delta d - \sum_{ij}\varphi_{ij} d_i d_j\big] \\
&+ (\psi')^3 \Delta d \\
&+ \psi''\big[1+|\nabla\varphi|^2 - \big(\sum_i \varphi_i d_i\big)^2\big],
\end{aligned}
$$

where we have used identities

$$
\begin{aligned}
&\sum_i d_i d_{ij} = 0 \ \forall j, \\
&\psi''(\psi')^2 - \sum_{ij} \psi''(\psi')^2 d_i^2 d_j^2 = 0, \\
&2\psi''\psi' \sum_i \varphi_i d_i - \psi''\psi' \sum_{ij} d_i d_j (\varphi_i d_i + \varphi_j d_j) = 0, \\
&\sum_{ij} \varphi_i\varphi_j d_i d_j = \big(\sum_i \varphi_i d_i\big)^2.
\end{aligned}
$$

Since φ and d are C^2-smooth in $\bar\Sigma_R$ and $\psi' \geq 1$, we have an estimate

$$
\begin{aligned}
&(1+|\nabla\varphi|^2)\Delta\varphi - \sum_{ij} \varphi_i\varphi_j\varphi_{ij} \\
&+ \psi'\big[2\sum_i \varphi_i d_i \Delta\varphi + (1+|\nabla\varphi|^2)\Delta d - 2\sum_{ij} d_i\varphi_j\varphi_{ij} - \sum_{ij}\varphi_i\varphi_j d_{ij}\big] \\
&+ (\psi')^2\big[\Delta\varphi + 2\sum_i \varphi_i d_i \Delta d - \sum_{ij}\varphi_{ij} d_i d_j\big] \\
&\leq c_0(\psi')^2
\end{aligned}
$$

for some constant c_0. On the other hand,

$$
1+|\nabla\varphi|^2 - \big(\sum_i \varphi_i d_i\big)^2 \geq 1,
$$

$\psi'' \leq 0$, and $\Delta d \leq 0$, and therefore

$$
\mathcal{E}(v) \leq c_0(\psi')^2 + (\psi')^3\Delta d + \psi'' \leq c_0(\psi')^2 + \psi'' \tag{6.26}
$$

in Σ_R. Choosing

$$\psi(t) = \frac{1}{c_0}\log(1+\beta t)$$

for some constant $\beta > 0$, we get

$$\psi'(t) = \frac{\beta}{c_0}(1+\beta t)^{-1}, \quad \psi''(t) = -\frac{\beta^2}{c_0}(1+\beta t)^{-2},$$

and

$$c_0(\psi')^2 + \psi'' = c_0\frac{\beta^2}{c_0^2}(1+\beta t)^{-2} - \frac{\beta^2}{c_0}(1+\beta t)^{-2} = 0. \tag{6.27}$$

Thus we need to find β and R so that $\psi'(t) \geq 1$ and $\psi(R) \geq L$. Choosing $R = \beta^{-1/2}$, with β large enough, we obtain

$$\psi'(t) \geq \frac{\beta}{c_0}\frac{1}{1+\beta R} = \frac{\beta}{c_0(1+\sqrt{\beta})} \geq 1 \quad \text{and} \quad \psi(R) = \frac{1}{c_0}\log(1+\sqrt{\beta}) \geq L.$$

By (6.26) and (6.27), v is a superminimizer in Σ_R, and therefore an upper barrier relative to $\varphi|\partial\Omega$.

Now it is easy to obtain a lower barrier relative to $\varphi|\partial\Omega$. Indeed, by the above, there exists an upper barrier $\tilde{u}$ relative to $-\varphi|\partial\Omega$. Then $v^- = -\tilde{u}$ is a lower barrier relative to $\varphi|\partial\Omega$.

We have proved:

Theorem 6.28. *Suppose that $\Omega \subset \mathbb{R}^n$ is a C^2-smooth bounded open set whose boundary $\partial\Omega$ has non-negative mean curvature with respect to inwards pointing normal. Then for each $\varphi \in C^2(\bar{\Omega})$ there exists a unique minimizer $u \in \operatorname{Lip}(\bar{\Omega})$ of the volume functional in $\mathcal{A} = \{v \in \operatorname{Lip}(\bar{\Omega})\colon v = \varphi \text{ on } \partial\Omega\}$.*

Proof. We have showed above that there are upper and lower barriers (for $\mathcal{V}$) relative to $\varphi|\partial\Omega$. By Theorem 6.19, the volume functional has a minimizer u in $\mathcal{A} = \{v \in \operatorname{Lip}(\bar{\Omega})\colon v = \varphi \text{ on } \partial\Omega\}$. The uniqueness follows from the Maximum principle □

6.29 Boundary mean curvature and the distance function

Let $\Omega \subset \mathbb{R}^n$ and suppose that $\partial\Omega$ is C^2. For $y \in \partial\Omega$ let N_y be the unit inner normal and $T_y = T_y\partial\Omega$ the tangent space to $\partial\Omega$ at y. Fix $y_0 \in \partial\Omega$. By rotating and translating we may assume that (a fixed) $y_0 = 0 \in \mathbb{R}^{n-1}$, $T_{y_0} = \mathbb{R}^{n-1}$, and $N_{y_0} = e_n$. Then in a neighborhood (in $\mathbb{R}^n$) U of 0, $\partial\Omega$ is given as a graph of a C^2-function

$$\varphi\colon \underbrace{T_0 \cap U}_{=U\cap\mathbb{R}^{n-1}} \to \mathbb{R},$$
$$\partial\Omega \cap U = \big\{\big(\underbrace{x_1,\dots,x_{n-1}}_{=:\tilde{x}}, \varphi(\tilde{x})\big) \colon \tilde{x} \in T_0 \cap U\big\}$$

Since $\mathbb{R}^{n-1}$ is assumed to be tangent to $\partial\Omega$ at 0, we have $\nabla\varphi(0) = 0$. For $y = \big(\tilde{y}, \varphi(\tilde{y})\big) \in \partial\Omega \cap U$, the inner unit normal is $N_y = (N_{y1}, N_{y2}, \dots, N_{yn})$, where

$$N_{yi} = \frac{-\varphi_i(\tilde{y})}{\sqrt{1+|\nabla\varphi(\tilde{y})|^2}}, \quad i = 1,\dots,n-1,$$
$$N_{yn} = \frac{1}{\sqrt{1+|\nabla\varphi(\tilde{y})|^2}},$$

where $\varphi_i = \varphi_{x_i}$. By rotating $\mathbb{R}^{n-1}$ around the x_n-axis, we may assume that the Hessian matrix of φ at 0 is the diagonal matrix

$$\varphi_{ij}(0) = \begin{pmatrix} \kappa_1 & 0 & \cdots & 0 & 0 \\ 0 & \kappa_2 & 0 & \vdots & \vdots \\ \vdots & 0 & \ddots & \vdots & 0 \\ 0 & \cdots & \cdots & \kappa_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & \kappa_{n-1} \end{pmatrix}.$$

Indeed, the Hessian matrix is symmetric and therefore it has real eigenvalues $\kappa_1, \ldots, \kappa_{n-1}$, with orthonormal eigenvectors. So, in these, so-called *principal coordinates*,

$$\varphi_{ij}(0) = \delta_{ij}\kappa_i.$$

In Section 4.2 we computed (in dimension 3) the mean curvature vector of a graph as

$$H = \operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) N,$$

where N is the upwards pointing (inner) unit normal. So,

$$H = \left(\frac{\Delta\varphi}{\sqrt{1+|\nabla\varphi|^2}} + \nabla\big(1+|\nabla\varphi|^2\big)^{-1/2}\cdot\nabla\varphi\right) N,$$

and therefore at $y_0 = 0 \in \partial\Omega$ (where $\nabla\varphi(0) = 0$) we have

$$H = \Delta\varphi(0) N_0.$$

The scalar mean curvature is therefore

$$\Delta\varphi(0) = \operatorname{tr}\big(\varphi_{ij}(0)\big) = \kappa_1 + \kappa_2 + \cdots + \kappa_{n-1}.$$

Lemma 6.30. *Let $\Omega \subset \mathbb{R}^n$ be bounded and C^k, $k \geq 2$. Then there exists $t_0 > 0$ such that*

$$d \in C^k\big(\bar{\Sigma}_{t_0}\big).$$

Proof. Since $\partial\Omega$ is compact and $C^k, k \geq 2$, $\partial\Omega$ satisfies a *uniform interior sphere condition*: for every $y_0 \in \partial\Omega$ there exists an open ball B_r of radius r such that $\bar{B}_r \cap (\mathbb{R}^n \backslash \Omega) = \{y_0\}$ and $r \geq R_0 > 0$, with R_0 independent of $y_0 \in \partial\Omega$. Indeed, again we may assume that $y_0 = 0 \in \mathbb{R}^{n-1}$ and $\partial\Omega$ is given as a graph of a C^k-function near y_0. The eigenvalues of the Hessian $\varphi_{ij}(0)$ are bounded from above by a constant that is independent of $y_0 \in \partial\Omega$. The inverse of this bound bounds R_0 from below. Let $t_0 < R_0$. For each $x \in \bar{\Sigma}_{t_0}$, there exits a unique $y(x) \in \partial\Omega$ such that $|x - y(x)| = d(x)$. To see this, take a closed ball $\bar{B}(x, d(x))$. Then $\bar{B}(x, d(x)) \cap \partial\Omega = \{y(x)\}$. The points $x \in \bar{\Sigma}_{t_0}$ and $y(x)$ are related by

$$x = y(x) + d(x)N_{y(x)}, \tag{6.31}$$

where $N_{y(x)}$ is the inner unit normal to $\partial\Omega$ at $y(x)$. We claim first that this relation determines $y = y(x)$ and $d = d(x)$ as C^{k-1}-functions of x. For a fixed $x_0 \in \bar{\Sigma}_{t_0}$, write $y_0 = y(x_0)$ and assume (after rotation and translation) that $y_0 = 0 \in \mathbb{R}^{n-1}$, $T_0 = T_{y_0}\partial\Omega = \mathbb{R}^{n-1}$, and that $\partial\Omega$ is given as a graph of a C^k-function $\varphi\colon V \to \mathbb{R}$, $V \subset \mathbb{R}^{n-1}$ a neighborhood of 0. Near 0 we may write $y \in \partial\Omega$ as $y = (\tilde{y}, \varphi(\tilde{y}))$. Define a mapping $g\colon V \times \mathbb{R} \to \mathbb{R}^n$,

$$g(\tilde{y}, t) = y + tN_y, \quad y = (\tilde{y}, \varphi(\tilde{y})).$$

Since $N_y = (N_{y1}, N_{y2}, \dots, N_{yn})$ is given by

$$\begin{aligned} N_{yi} &= \frac{-\varphi_i(\tilde{y})}{\sqrt{1 + |\nabla\varphi(\tilde{y})|^2}}, \quad i = 1, \dots, n-1, \\ N_{yn} &= \frac{1}{\sqrt{1 + |\nabla\varphi(\tilde{y})|^2}}, \end{aligned}$$

we see that g is C^{k-1}. We want to apply the inverse mapping theorem to g in a neighborhood of $(\tilde{y}, t) = (y_0, t) = (0, t)$, $t \le t_0$. In the principal coordinates the Jacobian $n \times n$-matrix of g at (y_0, t) is the diagonal matrix

$$\begin{pmatrix} 1 - t\kappa_1 & 0 & \cdots & 0 & 0 \\ 0 & 1 - t\kappa_2 & 0 & \vdots & \vdots \\ \vdots & 0 & \ddots & \vdots & 0 \\ 0 & \cdots & \cdots & 1 - t\kappa_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Hence the Jacobian determinant of g at $(y_0, d(x_0))$, with $d(x_0) \le t_0 < R_0$, is

$$\big(1 - \kappa_1 d(x_0)\big)\big(1 - \kappa_2 d(x_0)\big) \cdots \big(1 - \kappa_{n-1} d(x_0)\big) > 0$$

since every term $1 - \kappa_i d(x_0)$ is positive because $d(x_0) < R_0$ and therefore $\kappa_i d(x_0) < 1$. By the inverse mapping theorem $\tilde{y}_0$ and also $y_0 = (\tilde{y}_0, \varphi(\tilde{y}_0))$, and $d(x_0)$ depend C^{k-1}-smoothly on x_0.

On the other hand, since

$$\nabla d(x) = N_{y(x)}, \quad x \in \bar{\Sigma}_{t_0}$$

and the mappings $x \mapsto \tilde{y}(x)$ and $\tilde{y} \mapsto N_y$ are C^{k-1}-smooth, we conclude that $\nabla d(x)$ depends C^{k-1}-smoothly on x, and therefore $d \in C^k(\bar{\Sigma}_{t_0})$. □

Lemma 6.32. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded, and C^2-smooth set, whose boundary $\partial\Omega$ has non-negative mean curvature with respect to inner normal. Let d and Σ_{t_0} be given by Lemma 6.30. Then in $\bar{\Sigma}_{t_0}$*

(a) $\Delta d(x)$ decreases when x moves inwards along a normal to $\partial\Omega$;

(b) $\Delta d(x) \le 0$.

Proof. Fix $x_0 \in \bar{\Sigma}_{t_0}$. After a translation and rotations, we may assume that $y_0 = y(x_0) = 0$ and that $\partial\Omega$ is given as a graph of a C^2-function φ near y_0. In what follows we use principal coordinates at $\tilde{y}_0 = 0$. Since

$$\nabla d(x_0) = N_{\tilde{y}_0} = (0, 0, \dots, 0, 1),$$

we have

$$d_{in} = \frac{\partial}{\partial x_n} d_i = 0, \quad \forall i = 1, \dots, n-1.$$

Recall that

$$N_{yi} = \frac{-\varphi_i(\tilde{y})}{\sqrt{1+|\nabla\varphi(\tilde{y})|^2}}, \quad i = 1,\dots,n-1,$$
$$N_{yn} = \frac{1}{\sqrt{1+|\nabla\varphi(\tilde{y})|^2}},$$

so that

$$D_j N_{yi} = \frac{\partial}{\partial x_j} N_{yi} = -\kappa_i \delta_{ij}, \quad i = 1,\dots,n-1.$$

By the chain rule,

$$\begin{aligned}
d_{ij}(x_0) &= \big(D_j(\underbrace{D_i d}_{=N_i})\big)(x_0) = D_j N_i \circ y(x_0) \\
&= \sum_k D_k N_i(y_0) D_j y_k(x_0) \\
&= \sum_k (-\kappa_i \delta_{ik}) \left(\frac{\delta_{jk}}{1-\kappa_j d(x_0)} \right) \\
&= \frac{-\kappa_i \delta_{ij}}{1-\kappa_i d(x_0)}.
\end{aligned}$$

Hence the trace of (d_{ij}) is

$$\Delta d = \sum_{i=1}^{n-1} \frac{-\kappa_i}{1-\kappa_i d}.$$

Finally, when x moves inwards along a normal to $\partial\Omega$, $d(x)$ increases. We have two cases:

(i) if $\kappa_i \geq 0$, then $1-\kappa_d$ decreases and therefore

$$\frac{-\kappa_i}{1-\kappa_i d}$$

decreases;

(ii) if $\kappa_i \leq 0$, then $1-\kappa_d$ increases and therefore

$$\frac{-\kappa_i}{1-\kappa_i d}$$

decreases.

Alltogether $\Delta d(x)$ decreases when x moves inwards along a normal to $\partial\Omega$. This proves (a). On the other hand, $d(x) = 0$ on $\partial\Omega$, and so

$$\begin{aligned}
\Delta d(x) &= -\sum_i^{n-1} \kappa_i(x) \\
&= -\text{ the scalar mean curvature of } \partial\Omega \text{ at } x \in \partial\Omega \\
&\leq 0,
\end{aligned}$$

where $\kappa_i(x)$ are the principal curvatures of $\partial\Omega$ at x. Combining this with (a), proves the claim (b). $\square$

6.33 Regularity of the minimizer in Theorem 6.28

In this section we discuss briefly about the regularity of the minimizer $u \in \mathrm{Lip}(\bar{\Omega})$ of the the volume functional obtained in Theorem 6.28. The function u is a weak solution of the equation

$$-\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \tag{6.34}$$

in Ω, that is,

$$\int_\Omega \frac{\nabla u \cdot \nabla \eta}{\sqrt{1+|\nabla u|^2}} = 0$$

for all $\eta \in \mathrm{Lip}(\bar{\Omega})$, with $\eta|\partial\Omega = 0$. We write (6.34) as

$$-\operatorname{div} T(\nabla u) = \sum_{i=1}^n \frac{\partial T_i(\nabla u)}{\partial x_i} = 0,$$

where $T = (T_1, T_2, \ldots, T_n)\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$T_i(z) = \frac{z_i}{\sqrt{1+|z|^2}} = z_i \left(1 + \sum_{k=1}^n z_k^2\right)^{-1/2}.$$

Now

$$T_{ij}(z) := \frac{\partial T_i}{\partial z_j}(z) = \frac{1}{\sqrt{1+|z|^2}} \left(\delta_{ij} - \frac{z_i z_j}{1+|z|^2}\right).$$

Hence we can write (6.34) as

$$-\sum_{i,j=1}^n \frac{1}{\sqrt{1+|\nabla u|^2}} \left(\delta_{ij} - \frac{\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}}{1+|\nabla u|^2}\right) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0,$$

or equivalently,

$$-\sum_{i,j=1}^n T_{ij}(\nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \tag{6.35}$$

that is,

$$\int_\Omega \sum_{ij} T_{ij}(\nabla u) \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_j} = 0.$$

Now

$$\lambda(|z|)|\xi|^2 \le \sum_{ij} T_{ij}(z)\xi_i\xi_j \le \Lambda(|z|)|\xi|^2, \quad \xi \in \mathbb{R}^n, \tag{6.36}$$

with $0 < \lambda(|z|) \le \Lambda(|z|) < \infty$ for every $z \in \mathbb{R}^n$. Such an equation is a quasilinear elliptic equation, but it is not uniformly elliptic since the left-side of (6.36) tends to zero as $|z| \to \infty$. Moreover, the functions $T_{ij},\ z \mapsto T_{ij}(z)$ are C^∞. Since the solution u is (globally) Lipschitz in $\bar{\Omega}$, its gradient is bounded from above

$$|\nabla u| \le L < \infty \quad \text{a.e.} \tag{6.37}$$

"Freezing" the coefficients as

$$A_{ij}(x) := T_{ij}\big(\nabla u(x)\big)$$

and taking into account (6.37) we may interpret u as a solution of a *linear* uniformly elliptic second order PDE

$$\sum_{i,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0,$$

$$0 < \lambda|\xi|^2 \leq \sum_{ij} A_{ij}(z)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall z, \xi \in \mathbb{R}^n.$$

It follows then from the general theory that $u \in C^\infty(\Omega)$; see [G] for an overview.

6.38 Interior gradient estimate

Let us consider an n-dimensional C^2-smooth hypersurface $\Sigma \subset \mathbb{R}^{n+1}$. Assume that Σ is given as a level surface of a C^2-function $\Phi\colon U \to \mathbb{R}$, with $U \subset \mathbb{R}^{n+1}$ open, such that

$$\Sigma = \{x \in \mathbb{R}^{n+1} \colon \Phi(x) = 0\} \quad \text{and} \quad \nabla\Phi(x) \neq 0 \ \forall x \in \Sigma.$$

The unit normal field to Σ is (up to the choice of direction)

$$N = \frac{\nabla\Phi}{|\nabla\Phi|}.$$

With this choice, N points to the direction where Φ grows. For example, if Σ is the graph of a C^2-function $u\colon \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, then we can take

$$\Phi(x) = x_{n+1} - u(\tilde{x}), \quad x = (\tilde{x}, x_{n+1}) \in U := \Omega \times \mathbb{R}$$

since then

$$\nabla\Phi(x) = (-\underbrace{\nabla u(\tilde{x})}_{\in\mathbb{R}^n}, 1), \quad |\nabla\Phi| = \sqrt{1 + |\nabla u|^2}.$$

Let $g \in C^1(U)$. Then the *tangential gradient* δg of g on Σ is defined as the tangential component of ∇g, that is,

$$\delta g = \nabla g - (\nabla g \cdot N)N.$$

It is the orthogonal projection of $\nabla g(x)$ to $T_x\Sigma$. In our earlier notation $\delta g = \nabla^\Sigma g$. Its components are

$$\delta_i g = \frac{\partial g}{\partial x_i} - N_i \sum_{j=1}^{n+1} N_j \frac{\partial g}{\partial x_j}, \ \ i = 1, \ldots, n+1.$$

We also write

$$\delta_i = D_i - N_i \sum_j N_j D_j.$$

We have

$$\begin{aligned} N \cdot \delta g &\equiv 0, \\ |\delta g|^2 &= |\nabla g|^2 - (N \cdot \nabla g)^2, \ \text{ so} \\ |\delta g| &\leq |\nabla g|. \end{aligned}$$

Let $\kappa_1, \dots, \kappa_n$ be the principal curvatures of Σ at $y_0 \in \Sigma$ (with respect to N). Then (in principal coordinates)

$$\frac{\partial}{\partial x_j} \underbrace{\left(\frac{\frac{\partial}{\partial x_i}\Phi}{|\nabla \Phi|} \right)}_{=N_i} = -\kappa_i \delta_{ij}, \ i, j = 1, \dots, n;$$

$$\frac{\partial}{\partial x_j} \underbrace{\left(\frac{\frac{\partial}{\partial x_{n+1}}\Phi}{|\nabla \Phi|} \right)}_{=N_{n+1}} = 0, \ j = 1, \dots, n+1.$$

Hence

$$\begin{aligned}
\sum_{i=1}^{n+1} \delta_i N_i &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} N_i + \underbrace{\frac{\partial}{\partial x_{n+1}} N_{n+1}}_{=0} - \sum_{i=1}^{n+1} N_i \sum_{j=1}^{n+1} N_j \frac{\partial}{\partial x_j} N_i \\
&= -\sum_{i=1}^{n} \kappa_i - \sum_{i,j=1}^{n+1} N_i N_j \frac{\partial}{\partial x_j} N_i \\
&= -nH - \underbrace{\sum_{j=1} \tfrac{1}{2} N_j \frac{\partial}{\partial x_j} |N|^2}_{=0} \\
&= -nH.
\end{aligned}$$

We define (recall) the Laplace-Beltrami operator on Σ as

$$\Delta^\Sigma = \sum_{i=1}^{n+1} \delta_i \delta_i. \quad \text{(Check this!)}$$

We aim at proving the following:

Theorem 6.39. *Suppose that $u \in C^2(\Omega)$ is a solution of the minimal graph equation*

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0$$

in an open set $\Omega \subset \mathbb{R}^n$. Then there exist positive constants C_1 and C_2 depending only on n such that for every $x_0 \in \Omega$

$$|\nabla u(x_0)| \le C_1 \exp\left(C_2 \frac{\sup_\Omega u - u(x_0)}{\operatorname{dist}(x_0, \partial\Omega)} \right). \tag{6.40}$$

We need the following lemma. Since the graph $\Sigma = \Gamma_u$ is minimal, we have

$$\sum_{i=1}^{n+1} \delta_i N_i = 0.$$

We define $\omega \colon \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\omega(x, t) = \log \sqrt{1 + |\nabla u(x)|^2}.$$

Note that $\omega = -\log N_{n+1}$. We have (Exerc.)

$$\Delta^\Sigma \omega = \sum_{i=1}^{n+1} \delta_i \delta_i \omega \geq |\delta \omega|^2 \geq 0,$$

and therefore ω is subharmonic on Σ. The following says that ω satisfies a meanvalue inequality.

Lemma 6.41. *Let ω be as above. Then there exists a constant c_n depending only on n such that, for all $x_0 \in \Omega$, $0 < R < \operatorname{dist}(x_0, \partial\Omega)$, and $p = \big(x_0, u(x_0)\big)$, we have*

$$\omega(x_0) \leq \frac{c_n}{R^n} \int_{\Sigma_R(p)} \omega, \tag{6.42}$$

where $\Sigma_R(p) = \{q \in \Sigma \colon |p - q| < R\}$.

Proof. We prove the lemma in case $n \geq 3$. The 2-dimensional case is left as an extra exercise. We may assume that $p = 0$ ($\in \mathbb{R}^{n+1}$). For each $0 < \varepsilon < R$ and $x \in \mathbb{R}^{n+1}$ define

$$\varphi_\varepsilon(z) = \begin{cases} \frac{1}{2(n-2)}\big(\varepsilon^{2-n} - R^{2-n}\big) + \frac{1}{2n}\big(R^{-n} - \varepsilon^{-n}\big)|z|^2, & \text{if } 0 \leq |z| < \varepsilon; \\ \frac{|z|^{2-n}}{n(n-2)} + \frac{1}{2n}|z|^2 R^{-n} - \frac{1}{2(n-2)} R^{2-n}, & \text{if } \varepsilon \leq |z| \leq R; \\ 0 & \text{if } |z| > R. \end{cases}$$

Since $\varphi_\varepsilon \geq 0$ and both φ_ε and $\nabla \varphi_\varepsilon$ vanish on $\partial\Omega \times \mathbb{R}$, we get by integration by parts that

$$\int_\Sigma \omega \Delta^\Sigma \varphi_\varepsilon = \int_\Sigma \varphi_\varepsilon \Delta^\Sigma \omega \geq \int_\Sigma \varphi_\varepsilon |\delta \omega|^2 \geq 0.$$

We obtain by a direct computation that

$$\Delta^\Sigma |z|^\alpha = \alpha(\alpha - 2)|z|^{\alpha-2}\left(1 - \tfrac{z \cdot N}{|z|^2}\right) + \alpha n |z|^{\alpha-2},$$

and therefore

$$\Delta^\Sigma \varphi_\varepsilon = \begin{cases} R^{-n} - \varepsilon^{-n}, & \text{if } 0 < |z| < \varepsilon; \\ R^{-n} - |z|^{-2-n}(z \cdot N)^2, & \text{if } \varepsilon < |z| < R; \\ 0 & \text{if } |z| > R. \end{cases}$$

Hence

$$\begin{aligned} 0 &\leq \int_{\Sigma_\varepsilon(0)} (R^{-n} - \varepsilon^{-n})\omega + \int_{\Sigma_R(0) \setminus \Sigma_\varepsilon(0)} \big(R^{-n} - |z|^{-2-n}(z \cdot N)^2\big)\omega \\ &\leq \int_{\Sigma_\varepsilon(0)} R^{-n}\omega - \int_{\Sigma_\varepsilon(0)} \varepsilon^{-n}\omega + \int_{\Sigma_R(0) \setminus \Sigma_\varepsilon(0)} R^{-n}\omega - \int_{\Sigma_R(0) \setminus \Sigma_\varepsilon(0)} |z|^{-2-n}(z \cdot N)^2 \omega \\ &\leq \frac{1}{R^n} \int_{\Sigma_R(0)} \omega - \frac{1}{\varepsilon^n} \int_{\Sigma_\varepsilon(0)} \omega. \end{aligned}$$

Letting $\varepsilon \to 0$ and applying the Lebesgue differentiation theorem, we obtain

$$\omega(0) \leq \frac{c_n}{R^n} \int_{\Sigma_R(0)} w.$$

$\square$

Proof of Theorem 6.39. We may assume that $x_0 = 0 \in \Omega$ and $u(0) = 0$. Denote $B(r) = B^n(0,r)$ and $C(r) = B_r \times \mathbb{R}$. Throughout the proof c denotes a constant depending only on n and its value may change even within a line. By Lemma 6.41, we have

(6.43)
$$\omega(0) \leq \frac{c_n}{R^n} \int\limits_{\{x\in\mathbb{R}^n :\, |x|^2+|u(x)|^2\leq R^2\}} \omega\sqrt{1+|\nabla u|^2}dx \leq \frac{c_n}{R^n} \int\limits_{\{x\in\mathbb{R}^n :\, |x|\leq R, |u(x)|\leq R\}} \omega\sqrt{1+|\nabla u|^2}dx.$$

Let $0 < R < \frac{1}{3}\operatorname{dist}(0,\partial\Omega)$ and define

$$u_R = \begin{cases} 2R, & \text{if } u \geq R; \\ u+R, & \text{if } |u| \leq R; \\ 0 & \text{if } u \leq -R. \end{cases}$$

Choose $\eta \in C_0^1\big(B(2R)\big)$ such that $0 \leq \eta \leq 1,\ \eta|B(R) \equiv 1$, and $|\nabla\eta| \leq 2/R$. Use $\varphi = \omega u_R \eta$ as a test function in the minimal graph equation

$$0 = \int_\Omega \frac{\nabla u\cdot\nabla\varphi}{\sqrt{1+|\nabla u|^2}} = \int_\Omega \frac{\omega u_R \nabla u\cdot\nabla\eta}{\sqrt{1+|\nabla u|^2}} + \int_\Omega \frac{\omega\eta\nabla u\cdot\nabla u_R}{\sqrt{1+|\nabla u|^2}} + \int_\Omega \frac{u_R\eta\nabla u\cdot\nabla\omega}{\sqrt{1+|\nabla u|^2}}$$

to get

$$\int\limits_{\{x\in\mathbb{R}^n :\, |x|\leq R, |u(x)|\leq R\}} \frac{\omega|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \leq 2R \int\limits_{\{x\in\mathbb{R}^n :\, |x|\leq 2R, u>-R\}} \big(\omega|\nabla\eta| + \eta|\nabla\omega|\big). \tag{6.44}$$

On the other hand, since $\Delta^\Sigma\omega \geq 0$, we get, for all $\phi \in C_0^1\big(C(2R)\big)$, by integration by parts that

$$\begin{aligned}\int_{\Sigma\cap C(2R)} \phi^2|\delta\omega|^2 &\leq \int_{\Sigma\cap C(2R)} \phi^2\Delta^\Sigma\omega = -2\int_{\Sigma\cap C(2R)} \phi\delta\omega\cdot\delta\phi \\ &\leq 2\int_{\Sigma\cap C(2R)} \phi|\delta\omega||\delta\phi| \leq \tfrac{1}{2}\int_{\Sigma\cap C(2R)} \phi^2|\delta\omega|^2 + 2\int_{\Sigma\cap C(2R)} |\delta\phi|^2.\end{aligned}$$

Hence

$$\int_{\Sigma\cap C(2R)} \phi^2|\delta\omega|^2 \leq 4\int_{\Sigma\cap C(2R)} |\delta\phi|^2.$$

By Hölder's inequality we further obtain

$$\int_{\Sigma\cap\operatorname{supp}\phi} \phi|\delta\omega| \leq c\max|\delta\phi|\operatorname{Vol}(\Sigma\cap\operatorname{supp}\phi). \tag{6.45}$$

Next choose ϕ of the form

$$\phi(x,y) = \eta(x)\tau(t),$$

where η is as before and $\tau \in C_0^1\big((-2R, R+\sup_{B(2R)} u)\big)$, with $0 \leq \tau \leq 1,\ \tau \equiv 1$ in $[-R, \sup_{B_{2R}} u]$, and $|\frac{d\tau}{dt}| \leq c/R$. Since $\frac{\partial\omega}{\partial x_{n+1}} \equiv 0$, we have

$$|\nabla\omega|N_{n+1} \leq |\delta\omega|. \quad \text{(Exerc.)}$$

Hence

$$
\begin{aligned}
\int\limits_{\{|x|\le 2R, u>-R\}} \eta|\nabla\omega|dx &= \int\limits_{\{|x|\le 2R, u>-R\}} \eta|\nabla\omega|N_{n+1}\sqrt{1+|\nabla u|^2}dx \\
&= \int\limits_{\{(x,u(x))\colon |x|\le 2R, u>-R\}} \eta\underbrace{|\nabla\omega|N_{n+1}}_{\le|\delta\omega|} \\
&\le \int\limits_{\{(x,u(x))\colon |x|\le 2R, u>-R\}} \eta|\delta\omega| \le \int_{\Sigma\cap C(2R)} \phi|\delta\omega| \\
&\le c\max|\delta\phi|\operatorname{Vol}(\Sigma\cap\operatorname{supp}\phi) \le \tfrac{c}{R}\operatorname{Vol}(\Sigma\cap\operatorname{supp}\phi) \\
&\le \tfrac{c}{R}\int\limits_{\{|x|\le 2R, u\ge -2R\}} \sqrt{1+|\nabla u|^2}.
\end{aligned}
\tag{6.46}
$$

On the other hand, $\omega = \log\sqrt{1+|\nabla u|^2} \le \sqrt{1+|\nabla u|^2}$, so

$$
\int\limits_{\{|x|\le 2R, u\ge -R\}} \omega|\nabla\eta| \le \int\limits_{\{|x|\le 2R, u\ge -R\}} |\nabla\eta|\sqrt{1+|\nabla u|^2} \le \tfrac{2}{R}\int\limits_{\{|x|\le 2R, u\ge -R\}} \sqrt{1+|\nabla u|^2}.
\tag{6.47}
$$

Combining (6.46) and (6.47) with (6.44) we get

$$
\int\limits_{\{|x|\le R, |u|\le R\}} \frac{\omega|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \le c\int\limits_{\{|x|\le 2R, u\ge -2R\}} \sqrt{1+|\nabla u|^2}.
\tag{6.48}
$$

Therefore, the right-side of (6.42) can be estimated as

$$
\begin{aligned}
\int\limits_{\{|x|\le R, |u|\le R\}} \omega\sqrt{1+|\nabla u|^2} &= \int\limits_{\{|x|\le R, |u|\le R\}} \overbrace{\frac{\omega}{\sqrt{1+|\nabla u|^2}}}^{\le 1} + \int\limits_{\{|x|\le R, |u|\le R\}} \frac{\omega|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \\
&\le \int\limits_{\{|x|\le R, |u|\le R\}} 1 + \int\limits_{\{|x|\le R, |u|\le R\}} \frac{\omega|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \\
&\le cR^n + c\int\limits_{\{|x|\le 2R, u\ge -2R\}} \sqrt{1+|\nabla u|^2}.
\end{aligned}
\tag{6.49}
$$

Next we use $\varphi = \eta\max(u+2R, 0)$, where $\eta \in C_0^1\big(B(3R)\big)$, $0 \le \eta \le 1$, $\eta|B(2R) \equiv 1$, and $|\nabla\eta| \le 2/R$, in the (weak form of the) minimal graph equation. In the set, where $u \ge -2R$, $\varphi = \eta(u+2R)$ and $\nabla\varphi = (u+2R)\nabla\eta + \eta\nabla u$, otherwise $\nabla\varphi = 0$. We get

$$
0 = \int\limits_{\{|x|\le 3R, u\ge -2R\}} \frac{\eta|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} + \int\limits_{\{|x|\le 3R, u\ge -2R\}} \frac{(u+2R)\nabla u\cdot\nabla\eta}{\sqrt{1+|\nabla u|^2}}.
$$

Then

$$\begin{aligned}
\int\limits_{\{|x|\le 2R, u\ge -2R\}} \sqrt{1+|\nabla u|^2} &= \int\limits_{\{|x|\le 2R, u\ge -2R\}} \frac{1+|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \\
&= \int\limits_{\{|x|\le 2R, u\ge -2R\}} \frac{1}{1+|\nabla u|^2} + \int\limits_{\{|x|\le 2R, u\ge -2R\}} \frac{|\nabla u|^2}{1+|\nabla u|^2} \\
&\le cR^n + \int\limits_{\{|x|\le 3R, u\ge -2R\}} \frac{\eta|\nabla u|^2}{1+|\nabla u|^2} \\
&\le cR^n + \int\limits_{\{|x|\le 3R, u\ge -2R\}} \frac{(u+2R)|\nabla u||\nabla \eta|}{1+|\nabla u|^2} \qquad (6.50)\\
&\le cR^n + \int\limits_{\{|x|\le 3R, u\ge -2R\}} (u+2R)\tfrac{c}{R} \\
&\le cR^n + cR^n\big(c + \tfrac{c}{R}\sup_{B(3R)} u\big) \\
&\le R^n\big(c + \tfrac{c}{R}\sup_{B(3R)} u\big).
\end{aligned}$$

From (6.43), (6.49), and (6.50) we get

$$\omega(0) \le c + \tfrac{c}{R}\sup_{B(3R)} u,$$

and therefore

$$\begin{aligned}
|\nabla u(0)| \le \sqrt{1+|\nabla u(0)|^2} &= \exp\omega(0) \\
&\le \exp\left(c + \tfrac{c}{R}\sup_{B(3R)} u\right) \\
&= C_1 \exp\Big(\frac{c\sup_{B(3R)} u}{R}\Big).
\end{aligned}$$

This holds for every $0 < R < \frac{1}{3}\operatorname{dist}(0,\partial\Omega)$. Since we assumed $u(0)=0$, we finally have

$$|\nabla u(0)| \le C_1 \exp\left(\frac{C_2 \sup_\Omega (u-u(0))}{\operatorname{dist}(0,\partial\Omega)}\right).$$

□

6.51 Dirichlet problem with continuous boundary data

In this section we apply the interior gradient estimate and the theory of uniformly elliptic equations to the Dirichlet problem with continuous boundary values.

Theorem 6.52. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2-smooth boundary of nonnegative mean curvature with respect to inwards pointing normal. Let $\varphi \in C(\partial\Omega)$. Then there exists a unique $u \in C^\infty(\Omega)\cap C(\bar\Omega)$ such that $u|\partial\Omega = \varphi$ and*

$$\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \qquad (6.53)$$

in Ω.

Proof. Let $\varphi_j \in C^2(\mathbb{R}^n)$ be a sequence such that $\varphi_j|\partial\Omega \to \varphi$ uniformly. For each j there exists $u_j \in C^\infty(\Omega)\cap C(\bar{\Omega})$ that solves the equation (6.53) in Ω with boundary values φ_j. By the Maximum Principle 6.15,

$$\sup_\Omega |u_j - u_i| = \sup_{\partial\Omega} |u_j - u_i| = \sup_{\partial\Omega} |\varphi_j - \varphi_i|.$$

Hence there exists a continuous function $u \in C(\bar{\Omega})$ such that $u_j \to u$ uniformly in $\bar{\Omega}$. Fix a compact set $K \subset \Omega$. By the interior gradient estimate (6.40),

$$\sup_K |\nabla u_j| \le L,$$

where the constant $L = L(K) < \infty$ is independent of j. The theory of uniformly elliptic PDEs implies that

$$\sup_K |D^s u_j| \le L(K, s)$$

for any partial derivatives D^s of order s. It follows, in particular, that $u_j \to u$ in $C^2_{\text{loc}}(\Omega)\cap C(\bar{\Omega})$, and therefore also u solves (6.53) in Ω. The uniqueness follows again from the Maximum Principle. □

7 Functions of bounded variation

In this section we describe another approach to the existence of a minimizer of the volume functional. We define functions of bounded variations, give some basic properties of them mostly without proofs, and refer to textbooks, like [EG] and [G], for details.

7.1 Definitions and basic properties

Definition 7.2. Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{\text{loc}}(\Omega)$. Define

$$\int_\Omega |Du| := \sup \left\{ \int_\Omega u \operatorname{div} g \colon g = (g_1, \ldots, g_n) \in C^1_0(\Omega; \mathbb{R}^n),\ |g| \le 1 \right\}.$$

Above $\int_\Omega |Du|$ should be understood just as a notation (not an integral). Furthermore,

$$\operatorname{div} g = \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}$$

is the usual divergence.

Examples 7.3. (a) If $u \in C^1(\Omega)$, then integration by parts implies that

$$\int_\Omega u \operatorname{div} g = -\int_\Omega \nabla u \cdot g \quad \forall g \in C^1_0(\Omega; \mathbb{R}^n),$$

and so

$$\int_\Omega |Du| = \int_\Omega |\nabla u|.$$

(b) More generally, if u belongs to the Sobolev space $W^{1,1}_{\text{loc}}(\Omega)$, then again

$$\int_\Omega |Du| = \int_\Omega |\nabla u|,$$

where ∇u is the distributional gradient of u.

Definition 7.4. A function $u \in L^1(\Omega)$ is said to have *bounded variation in* Ω if

$$\int_\Omega |Du| < \infty.$$

We denote by $\mathrm{BV}(\Omega)$ the vector space of all functions $u \in L^1(\Omega)$ with bounded variation in Ω.

Definition 7.5. Similarly, a function $u \in L^1_{\mathrm{loc}}(\Omega)$ has *locally bounded variation* and belongs to $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ if

$$\int_V |Du| < \infty$$

for every relatively compact open set $V \Subset \Omega$.

Theorem 7.6. *For every $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ there exists a Radon measure μ on Ω and a μ-measurable mapping $\sigma\colon \Omega \to \mathbb{R}^n$ such that*

(i) $|\sigma(x)| = 1$ *for μ-a.e. $x \in \Omega$;*

(ii)

$$\int_\Omega u \operatorname{div} g \, dx = -\int_\Omega g \cdot \sigma \, d\mu$$

for every $g \in C^1_0(\Omega; \mathbb{R}^n)$.

Proof. Let $L\colon C^1_0(\Omega; \mathbb{R}^n) \to \mathbb{R}$ be the linear functional

$$L(g) = -\int_\Omega u \operatorname{div} g \, dm.$$

Since $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, we have

$$\sup\{L(g)\colon g \in C^1_0(V; \mathbb{R}^n), |g| \le 1\} =: C(V) < \infty$$

for every relatively compact $V \Subset \Omega$. Hence

$$|L(g)| \le C(V)\|g\|_\infty \quad \forall g \in C^1_0(V, \mathbb{R}^n). \tag{7.7}$$

For each $g \in C_0(\Omega; \mathbb{R}^n)$ choose an open set V such that $\operatorname{supp} g \subset V \Subset \Omega$. Furthermore, let $g_k \in C^1_0(V; \mathbb{R}^n), k \in \mathbb{N}$, be a sequence such that $g_k \to g$ uniformly in V. Define

$$\bar{L}(g) = \lim_{k\to\infty} L(g_k).$$

By (7.7), the limit exists and is independent of the chosen sequence g_k (and of $V \supset \operatorname{supp} g$). Thus L uniquely extends to a linear functional

$$\bar{L}\colon C^1_0(\Omega; \mathbb{R}^n) \to \mathbb{R}$$

and

$$\sup\{\bar{L}(g)\colon g \in C_0(\Omega; \mathbb{R}^n), \operatorname{supp} g \subset K, |g| \le 1\} < \infty$$

for every compact $K \subset \Omega$. The claim then follows from the Riesz representation theorem. □

By Examples 7.3 (b), $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$. The converse inclusion does not hold. Indeed, suppose that $E \subset \mathbb{R}^n$ is a bounded open set, with C^2-smooth boundary. Let $\Omega \subset \mathbb{R}^n$ be an open set such that $\Omega \cap E \neq \emptyset \neq \Omega \cap (\mathbb{R}^n \setminus E)$. Furthermore, let χ_E be the characteristic function of E. Then $\chi_E \in L^1(\Omega)$ and for all $g \in C_0^1(\Omega; \mathbb{R}^n)$, $|g| \leq 1$,

$$\int_\Omega \chi_E \operatorname{div} g = \int_E \operatorname{div} g = \int_{\partial E} g \cdot \nu dH^{n-1} \leq H^{n-1}(\partial E \cap \Omega) < \infty. \tag{7.8}$$

Here H^{n-1} is the (normalized) $(n-1)$-dimensional Hausdorff measure. Hence $\chi_E \in \mathrm{BV}(\Omega)$. On the other hand, $\chi_E \notin W^{1,1}(\Omega)$ (Exerc.).

Definition 7.9. Let $E \subset \mathbb{R}^n$ be (Lebesgue) measurable and let $\Omega \subset \mathbb{R}^n$ be open. Then the *perimeter* of E in Ω is

$$P(E, \Omega) = \int_\Omega |D\chi_E|.$$

Furthermore, $E \subset \mathbb{R}^n$ has *finite perimeter in* Ω if $P(E, \Omega) < \infty$, i.e. $\chi_E \in \mathrm{BV}(\Omega)$. If a Borel set E has finite perimeter in every bounded open set $\Omega \subset \mathbb{R}^n$, then E is called a *Caccioppoli set.*

Remarks 7.10.

1. If $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, we denote by $\|Du\|$ the Radon measure μ given by Theorem 7.6 and by
$$[Du] = \|Du\| \llcorner \sigma$$
the vector valued measure $d[Du] = \sigma \, d\|Du\|$. Hence
$$\int_\Omega u \operatorname{div} g = -\int_\Omega g \cdot \sigma \, d\|Du\| = -\int_\Omega g \cdot d[Du]$$
for $g \in C_0^1(\Omega; \mathbb{R}^n)$.
2. If $u \in \mathrm{BV}(\Omega)$ and $V \Subset \Omega$ is an open subset, then
$$\|Du\|(V) = \sup\left\{\int_V u \operatorname{div} g \, dx \colon g \in C_0^1(\Omega; \mathbb{R}^n), |g| \leq 1\right\}.$$
Hence, using our earlier notation,
$$\int_V |Du| = \|Du\|(V).$$
3. If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set of locally finite perimeter in Ω and $u = \chi_E$, we also write
$$\|\partial E\| = \mu \quad \text{and} \quad \nu_E = -\sigma$$
for the measure μ and the mapping σ given by Theorem 7.6. Thus
$$\int_E \operatorname{div} g = \int_\Omega \chi_E \operatorname{div} g = \int_\Omega g \cdot \nu_E \, d\|\partial E\|$$
for all $g \in C_0^1(\Omega; \mathbb{R}^n)$ and
$$\|\partial E\|(V) = \sup\left\{\underbrace{\int_E \operatorname{div} g}_{\int_V \chi_E \operatorname{div} g} : g \in C_0^1(V; \mathbb{R}^n), |g| \leq 1\right\}$$
for $V \Subset \Omega$.

4. Furthermore, for $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, we write

$$\mu^i = \|Du\| \llcorner \sigma_i$$

for the signed Radon measure $d\mu^i = \sigma_i \, d\|Du\|$, and

$$\mu^i = \mu^i_a + \mu^i_s,$$

where $\mu^i_a \ll m_n$ and $\mu^i_s \perp m_n$ are the absolutely continuous and the singular part of μ^i (with respect to the Lebesgue measure m_n). Hence by the Radon-Nikodym theorem

$$\mu^i_a = m_n \llcorner u_i$$

for some $u_i \in L^1_{\mathrm{loc}}(\Omega)$. [In other words, $d\mu^i_a = u_i \, dm_n$, and $u_i = d\mu^i_a / dm_n$ is the Radon-Nikodym derivative of μ^i_a with respect to m_n.] We write

$$\begin{aligned} \frac{\partial u}{\partial x_i} &:= u_i, \quad i = 1, \ldots, n; \\ Du &:= (\tfrac{\partial u}{\partial x_1}, \ldots, \tfrac{\partial u}{\partial x_n}); \\ [Du]_a &:= (\mu^1_a, \ldots, \mu^n_a) = m_n \llcorner Du; \\ [Du]_s &:= (\mu^1_s, \ldots, \mu^n_s). \end{aligned}$$

So,

$$[Du] = [Du]_a + [Du]_s = m_n \llcorner Du + [Du]_s,$$

that is, $Du \in L^1_{\mathrm{loc}}(\Omega); \mathbb{R}^n)$ is the density of the absolutely continuous part of $[Du]$.

Examples 7.11. 1. If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, then $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\|Du\| = m_n \llcorner |\nabla u|$. Furthermore,

$$\sigma(x) = \begin{cases} \frac{\nabla u(x)}{|\nabla u(x)|}, & \text{if } \nabla u(x) \neq 0; \\ 0, & \text{if } \nabla u(x) = 0 \end{cases}$$

for m_n-a.e. $x \in \Omega$.

2. Suppose that E is a smooth open subset of $\mathbb{R}^n$, with $H^{n-1}(\partial E \cap K) < \infty$ for every compact $K \subset \Omega$. Then for every relatively compact open set $V \Subset \Omega$ and $g \in C^1_0(\mathbb{R}^n)$,

$$\int_E \operatorname{div} g \, dx = \int_{\partial E \cap V} g \cdot \nu \, dH_{n-1},$$

where ν is the outward pointing unit normal field to ∂E. If, furthermore, $|g| \leq 1$, we have

$$\int_E \operatorname{div} g \, dx = \int_V \chi_E \operatorname{div} g \, dx = \int_{\partial E \cap V} g \cdot \nu \, dH_{n-1} \leq H^{n-1}(\partial E \cap V) < \infty.$$

So, $\|\partial E\|(\Omega) = H^{n-1}(\partial E \cap \Omega)$ and $\nu_E = \nu \ H^{n-1}$-a.e. in $\partial E \cap \Omega$.

Remark 7.12. We noticed earlier that $W^{1,1}_{\mathrm{loc}}(\Omega) \subsetneq \mathrm{BV}_{\mathrm{loc}}(\Omega)$. In fact, a function $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ belongs to $W^{1,p}_{\mathrm{loc}}(\Omega)$, $p \geq 1$, if and only if

$$u \in L^p_{\mathrm{loc}}(\Omega), \ [Du]_s = 0, \ \text{and } Du \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n).$$

Theorem 7.13 (Lower semicontinuity). *Let $\Omega \subset \mathbb{R}^n$ be open and $u_j \in \mathrm{BV}(\Omega)$, $j \in \mathbb{N}$ such that $u_j \to u$ in $L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\int_\Omega |Du| \leq \liminf_{j\to\infty} \int_\Omega |Du_j|. \tag{7.14}$$

Proof. Let $g \in C^1_0(\Omega;\mathbb{R}^n)$, $|g| \leq 1$. Then

$$\begin{aligned}\left|\int_\Omega u \operatorname{div} g - \int_\Omega u_j \operatorname{div} g\right| &= \left|\int_\Omega (u-u_j)\operatorname{div} g\right| \\ &\leq \int_\Omega |u-u_j| \underbrace{|\operatorname{div} g|}_{\leq C} \\ &\leq C\int_{\operatorname{supp} g} |u-u_j| \to 0\end{aligned}$$

as $j \to \infty$. Thus for all $g \in C^1_0(\Omega;\mathbb{R}^n)$, $|g| \leq 1$, we have

$$\int_\Omega u \operatorname{div} g = \lim_{j\to\infty}\int_\Omega u_j \operatorname{div} g \leq \liminf_{j\to\infty}\int_\Omega |Du_j|$$

and (7.14) follows by taking the supremum of the left-side over all such g. □

Theorem 7.15. *Equipped with the* BV*-norm*

$$\|u\|_{\mathrm{BV}} := \|u\|_{L^1(\Omega)} + \int_\Omega |Du|$$

$\mathrm{BV}(\Omega)$ *is a Banach space.*

Proof. Exercise. □

Functions in Sobolev spaces $W^{1,p}(\Omega)$, $1 \leq p < \infty$, can be approximated by $C^\infty(\Omega)$ functions in the Sobolev norm

$$\|u\|_{1,p} := \|u\|_p + \||\nabla u|\|_p.$$

In fact, $W^{1,p}(\Omega)$ is the completion of $C^\infty(\Omega)$ in the Sobolev norm and since $\mathrm{BV}(\Omega) \neq W^{1,1}(\Omega)$, functions in $\mathrm{BV}(\Omega)$ can not be approximated in the BV-norm. However,

Theorem 7.16 (Approximation). *Let $u \in \mathrm{BV}(\Omega)$. Then there exists a sequence $u_j \in C^\infty(\Omega)$, $j \in \mathbb{N}$, such that*

$$\lim_{j\to\infty}\int_\Omega |u_j - u| = 0,$$

$$\lim_{j\to\infty}\int_\Omega |Du_j| = \int_\Omega |Du|.$$

Suppose that $u \in \mathrm{BV}(\Omega)$ and $u_j \in C^\infty(\Omega)$ are as above. For each $j \in \mathbb{N}$ let μ_j be the vector-valued Radon-measure defined by

$$\mu_j(B) = \int_{B\cap\Omega} \nabla u_j \, dx$$

for Borel sets $B \subset \mathbb{R}^n$. Furthermore, let μ be the vector-valued Radon measure

$$\mu(B) = \int_{B\cap\Omega} d[Du] = \int_{B\cap\Omega} \sigma \, d\|Du\|.$$

Then $\mu_j \rightharpoonup \mu$.

Theorem 7.17 (Compactness). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary $\partial\Omega$. Suppose that $u_j \in \mathrm{BV}(\Omega)$, $j \in \mathbb{N}$, is a sequence such that*

$$\sup_j \|u_j\|_{\mathrm{BV}} < \infty.$$

Then there exist a subsequence (u_{j_k}) and $u \in \mathrm{BV}(\Omega)$ such that $u_{j_k} \to u$ in $L^1(\Omega)$.

Proof. For $j = 1, 2, \ldots$, choose $g_j \in C^\infty(\Omega)$ such that

$$\int_\Omega |u_j - g_j| < 1/j \quad \text{and} \quad \sup_j \int_\Omega |\nabla g_j| < \infty.$$

Since $\partial\Omega$ is assumed to be Lipschitz, there exist $u \in L^1(\Omega)$ and a subsequence (g_{j_k}) such that $g_{j_k} \to u$ in $L^1(\Omega)$; see Theorem 1 and Remark in Section 4.6 in [EG]. Then also $u_{j_k} \to u$ in $L^1(\Omega)$, and by lower semicontinuity (Theorem 7.13) $u \in \mathrm{BV}(\Omega)$. □

Theorem 7.18 ("Existence of minimal surfaces"). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $L \subset \mathbb{R}^n$ be a Caccioppoli set. Then there exists a measurable set $E \subset \mathbb{R}^n$ such that*

$$E \setminus \Omega = L \setminus \Omega$$

and

$$\int_{\mathbb{R}^n} |D\chi_E| \le \int_{\mathbb{R}^n} |D\chi_F|$$

for every measurable $F \subset \mathbb{R}^n$, with

$$F \setminus \Omega = L \setminus \Omega.$$

Proof. Let $R > 0$ be so large that $\bar{\Omega} \subset B = B^n(0, R) = \{x \in \mathbb{R}^n \colon |x| < R\}$. Let $F \subset \mathbb{R}^n$ be a measurable set such that $F \setminus \Omega = L \setminus \Omega$. Then

$$\int_{\mathbb{R}^n} |D\chi_F| = \|\partial F\|(\mathbb{R}^n) = \|\partial F\|(B) + \|\partial F\|(\mathbb{R}^n \setminus B) = \int_B |D\chi_F| + \int_{\mathbb{R}^n \setminus B} |D\chi_L|.$$

Thus we need to find a measurable set $E \subset B$ such that $E = L$ in $B \setminus \Omega$ and

$$\int_B |D\chi_E| \le \int_B |D\chi_F| \tag{7.19}$$

for all measurable sets F, with $F = L$ in $B \setminus \Omega$. Let $\{E_j\}$ be a minimizing sequence of admissible measurable sets, i.e. $E_j = L$ in $B \setminus \Omega$ and

$$\int_B |D\chi_{E_j}| \to \inf \left\{ \int_B |D\chi_F| \colon F = L \text{ in } B \setminus \Omega \right\}.$$

Since the Caccioppoli set L itself is admissible, we may assume that

$$\sup_j \int_B |D\chi_{E_j}| < \infty.$$

Hence (χ_{E_j}) is a bounded sequence in $\mathrm{BV}(\Omega)$, and therefore there exists a subsequence, still denoted by (χ_{E_j}) and $f \in L^1(B)$ such that $\chi_{E_j} \to f$ in $L^1(B)$. Hence (for a new subsequence) $\chi_{E_j}(x) \to f(x)$ for a.e. $x \in B$. Since $\chi_{E_j}(x)$ is either 0 or 1, we may assume that f is the characteristic function of a measurable set E such that $E = L$ in $B \setminus \Omega$. Thus E is admissible and by lower semicontinuity (Theorem 7.13)

$$\int_B |D\chi_E| = \int_B |Df| \le \liminf_{j\to\infty} \int_B |D\chi_{E_j}| = \inf\left\{\int_B |D\chi_F|\colon F = L \text{ in } B\setminus\Omega\right\}.$$

□

Thus the perimeter of E, $P(E,\Omega)$, minimizes the perimeters of all sets with the same "boundary values" L. By recalling (7.8), we may say that, roughly speaking, ∂E minimizes the measure of all "surfaces" with boundary values $\partial L \cap \partial\Omega$. In order to apply this idea to the minimizing problem of the volume functional, we need to discuss about boundary values of BV-functions, i.e. *traces* of BV-functions. The idea is that if $\partial\Omega$ is sufficiently nice and $u \in \mathrm{BV}(\Omega)$, it is possible to define boundary values of u.

Suppose that Ω is the upper-half space

$$\Omega = \mathbb{R}^{n-1} \times \mathbb{R}_+ = \{(x,t)\colon x \in \mathbb{R}^{n-1}, t > 0\}.$$

For $y \in \mathbb{R}^{n-1}$ and $r > 0$ denote $C_r^+(y) = B^{n-1}(y,r) \times (0,r) \subset \bar\Omega$. Let $u \in \mathrm{BV}(C_R^+(0))$ for some $R > 0$. Then there exists $u^+ \in L^1(B^{n-1}(0,R); H^{n-1})$ such that

$$\lim_{r\to 0+} r^{-n} \int_{C_r^+(y)} |u(x) - u^+(y)| dm_n(x) = 0$$

for H^{n-1}-a.e. $y \in B^{n-1}(0,R)$. Note that this conclusion is much stronger than the one in the Lebesgue differentiation theorem. The function u^+ is called the *trace of* u and it is denoted by $\mathrm{Tr}\, u$. Obviously,

$$u^+(y) = \lim_{r\to 0+} \frac{1}{m_n\big(C_r^+(y)\big)} \int_{C_r^+(y)} u(x) dm_n(x) \quad \text{for } H^{n-1}\text{-a.e. } y \in B^{n-1}(0,R).$$

Note that it is not possible to well-define a trace of a general L^1-function. It is exactly the existence of "derivative" that makes BV-functions more regular than mere L^1-functions and, consequently, enables the definition of the trace.

More generally, suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary $\partial\Omega$. Then there exists a bounded linear mapping $\mathrm{Tr}\colon \mathrm{BV}(\Omega) \to L^1(\partial\Omega; H^{n-1})$ such that

$$\int_\Omega u \operatorname{div} g\, dx = -\int_\Omega g \cdot d[Du] + \int_{\partial\Omega} (g\cdot\nu)\, \mathrm{Tr}\, u\, dH^{n-1} \tag{7.20}$$

for all $u \in \mathrm{BV}(\Omega)$ and $g \in C^1(\mathbb{R}^n;\mathbb{R}^n)$, where ν is the outward pointing unit normal to $\partial\Omega$ (defined H^{n-1}-a.e. on $\partial\Omega$ since $\partial\Omega$ is Lipschitz). Note that g need not vanish near $\partial\Omega$. Moreover, for H^{n-1}-a.e. $x \in \partial\Omega$

$$\lim_{r\to 0+} \frac{1}{m\big(B^n(x,r)\cap\Omega\big)} \int_{B^n(x,r)\cap\Omega} |u(y) - \mathrm{Tr}\, u(x)|\, dm(y) = 0,$$

and so

$$\mathrm{Tr}\, u(x) = \lim_{r\to 0+} \frac{1}{m\big(B^n(x,r)\cap\Omega\big)} \int_{B^n(x,r)\cap\Omega} u(y)\, dm(y).$$

We formulate (see e.g. [EG]):

Theorem 7.21. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Then there exists a unique continuous linear operator* $\mathrm{Tr}\colon \mathrm{BV}(\Omega) \to L^1(\partial\Omega; H^{n-1})$ *such that for every $u \in C^\infty(\bar{\Omega})$ we have $\mathrm{Tr}\, u = u|\partial\Omega$. Moreover, the map* Tr *is surjective.*

We also have (see [G, p. 41]):

Theorem 7.22. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Given $\varphi \in L^1(\partial\Omega; H^{n-1})$ and $\varepsilon > 0$, there exist $f \in W^{1,1}(\Omega)$ and a constant $A = A(\partial\Omega)$ such that $\mathrm{Tr}\, f = \varphi$ and*

$$\int_\Omega |f| \le \varepsilon \|\varphi\|_{L^1(\partial\Omega)} \quad \textit{and}$$
$$\int_\Omega |Df| \le A\|\varphi\|_{L^1(\partial\Omega)}.$$

If, moreover, $\partial\Omega$ is C^1, the constant A can be taken as $A = 1+\varepsilon$.

Next we discuss about extensions of BV-functions.

Theorem 7.23. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary $\partial\Omega$. Let $f_1 \in \mathrm{BV}(\Omega)$ and $f_2 \in \mathrm{BV}(\mathbb{R}^n \setminus \bar{\Omega})$. Define $f\colon \mathbb{R}^n \setminus \partial\Omega \to \mathbb{R}$,*

$$f(x) = \begin{cases} f_1(x), & \textit{if } x \in \Omega; \\ f_2(x), & \textit{if } x \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

Then $f \in \mathrm{BV}(\mathbb{R}^n)$ and

$$\|Df\|(\mathbb{R}^n) = \|Df_1\|(\Omega) + \|Df_2\|(\mathbb{R}^n \setminus \bar{\Omega}) + \int_{\partial\Omega} |\mathrm{Tr}\, f_1 - \mathrm{Tr}\, f_2|\, dH^{n-1}. \tag{7.24}$$

Proof. Let $g \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$ with $|g| \le 1$. By (7.20), we have

$$\begin{aligned} \int_{\mathbb{R}^n} f \operatorname{div} g &= \int_\Omega f_1 \operatorname{div} g + \int_{\mathbb{R}^n \setminus \bar{\Omega}} f_2 \operatorname{div} g \\ &= -\int_\Omega g \cdot d[Df_1] - \int_{\mathbb{R}^n \setminus \bar{\Omega}} g \cdot d[Df_2] + \int_{\partial\Omega} (\mathrm{Tr}\, f_1 - \mathrm{Tr}\, f_2) g \cdot \nu\, dH^{n-1} \\ &\le \|Df_1\|(\Omega) + \|Df_2\|(\mathbb{R}^n \setminus \bar{\Omega}) + \int_{\partial\Omega} |\mathrm{Tr}\, f_1 - \mathrm{Tr}\, f_2|\, dH^{n-1}. \end{aligned}$$

It follows that $f \in \mathrm{BV}(\mathbb{R}^n)$ and $\|Df\|(\mathbb{R}^n)$ is at most the right-side of (7.24). Conversely,

$$\begin{aligned} \int_{\mathbb{R}^n} f \operatorname{div} g &= -\int_{\mathbb{R}^n} g \cdot d[Df] \\ &= -\int_\Omega g \cdot d[Df_1] - \int_{\mathbb{R}^n \setminus \bar{\Omega}} g \cdot d[Df_2] + \int_{\partial\Omega} (\mathrm{Tr}\, f_1 - \mathrm{Tr}\, f_2) g \cdot \nu\, dH^{n-1} \end{aligned}$$

for all $g \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$. Hence

$$[Df] = \begin{cases} [Df_1], & \text{in } \Omega; \\ [Df_2], & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \end{cases}$$

and

$$\begin{aligned}
&\int_{\Omega} g \cdot d[Df_1] + \int_{\mathbb{R}^n \setminus \bar{\Omega}} g \cdot d[Df_2] - \int_{\partial\Omega} (\operatorname{Tr} f_1 - \operatorname{Tr} f_2) g \cdot \nu \, dH^{n-1} \\
&= \int_{\mathbb{R}^n} g \cdot d[Df] \\
&= \int_{\Omega} g \cdot d[Df] + \int_{\mathbb{R}^n \setminus \bar{\Omega}} g \cdot d[Df] + \int_{\partial\Omega} g \cdot d[Df] \\
&= \int_{\Omega} g \cdot d[Df_1] + \int_{\mathbb{R}^n \setminus \bar{\Omega}} g \cdot d[Df_2] + \int_{\partial\Omega} g \cdot d[Df].
\end{aligned}$$

It follows that

$$\int_{\partial\Omega} g \cdot d[Df] = -\int_{\partial\Omega} (\operatorname{Tr} f_1 - \operatorname{Tr} f_2) g \cdot \nu \, dH^{n-1}$$

for all $g \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$. Hence

$$\|Df\|(\partial\Omega) = \int_{\partial\Omega} |\operatorname{Tr} f_1 - \operatorname{Tr} f_2| \, dH^{n-1},$$

and therefore

$$\begin{aligned}
\|Df\|(\mathbb{R}^n) &\le \|Df_1\|(\Omega) + \|Df_2\|(\mathbb{R}^n \setminus \bar{\Omega}) + \int_{\partial\Omega} |\operatorname{Tr} f_1 - \operatorname{Tr} f_2| \, dH^{n-1} \\
&= \|Df\|(\Omega) + \|Df\|(\mathbb{R}^n \setminus \bar{\Omega}) + \|Df\|(\partial\Omega) \\
&= \|Df\|(\mathbb{R}^n).
\end{aligned}$$

□

7.25 Volume functional among BV-functions

Let us return back to the volume functional and minimal graphs. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set and let $u\colon \Omega \to \mathbb{R}$ be a continuous function. We define the *relaxed area* (or *relaxed volume*) of its graph Γ_u as

$$\operatorname{Vol}(\Gamma_u) = \inf \left\{ \liminf_{k\to\infty} \mathcal{V}(u_k) \colon u_k \to u \text{ uniformly in } \Omega,\ u_k \in C^1(\bar{\Omega}) \right\}, \tag{7.26}$$

where the infimum is taken over all such sequences (u_k).

Remarks 7.27.

1. The relaxed area functional is lower semicontinuous with respect to uniform convergence.
2. The relaxed area of the graph of a Lipschitz function coincides with its usual area.

Next we extend the definition even more:

Definition 7.28. Let $u \in L^1(\Omega)$. We define

$$\mathcal{V}(u) = \inf \left\{ \liminf_{k\to\infty} \mathcal{V}(u_k) \colon u_k \to u \text{ in } L^1(\Omega),\ u_k \in C^1(\bar{\Omega}) \right\}. \tag{7.29}$$

It turns out that

$$\mathcal{V}(u) = \sup\left\{\int_\Omega \left(u\sum_{i=1}^n \tfrac{\partial g_i}{\partial x_i} + g_{n+1}\right) : g \in C_0^1(\Omega;\mathbb{R}^{n+1}),\ |g| \le 1\right\}. \tag{7.30}$$

We also write

$$\int_\Omega \sqrt{1+|Du|^2} := \mathcal{V}(u) \tag{7.31}$$

for functions $u \in L^1(\Omega)$. Thus for bounded open sets and functions $u \in L^1(\Omega)$

$$\int_\Omega \sqrt{1+|Du|^2} = \sup\left\{\int_\Omega \left(u\sum_{i=1}^n \tfrac{\partial g_i}{\partial x_i} + g_{n+1}\right) : g \in C_0^1(\Omega;\mathbb{R}^{n+1}),\ |g| \le 1\right\}. \tag{7.32}$$

Since

$$\int_\Omega |Du| \le \int_\Omega \sqrt{1+|Du|^2} \le \int_\Omega |Du| + m_n(\Omega), \tag{7.33}$$

we see that $\mathrm{BV}(\Omega)$-functions in a bounded open set Ω are exactly those functions whose graphs have finite relaxed area.

We notice that

$$\int_\Omega \sqrt{1+|Du|^2} = \int_\Omega \sqrt{1+|\nabla u|^2}$$

if $u \in C^1(\Omega)$, or more generally, if $u \in W^{1,1}(\Omega)$. The lower semicontinuity

$$\int_\Omega \sqrt{1+|Du|^2} \le \liminf_{j\to\infty} \int_\Omega \sqrt{1+|Du_j|^2}$$

holds whenever $u_j \to u$ in $L^1(\Omega)$.

Next we formulate a minimizing problem: Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary $\partial\Omega$. For $\varphi \in L^1(\partial\Omega; H^{n-1})$, denote

$$\mathcal{A} = \{u \in \mathrm{BV}(\Omega)\colon\ \mathrm{Tr}\, u = \varphi\}.$$

Problem: Find a minimizer for the relaxed area functional $\mathcal{V}$ among functions in $\mathcal{A}$.

Since the trace operator $\mathrm{Tr}\colon\ \mathrm{BV}(\Omega) \to L^1(\partial\Omega)$ is surjective, the set $\mathcal{A}$ of admissible functions is non-empty, and therefore, for all $v \in \mathcal{A}$

$$I := \inf\{\mathcal{V}(u)\colon u \in \mathcal{A}\} \le \int_\Omega |Dv| + m_n(\Omega) < \infty.$$

Thus we may find a minimizing sequence $u_j \in \mathcal{A},\ j \in \mathbb{N}$, with

$$\lim_{j\to\infty} \mathcal{V}(u_j) = I.$$

By compactness, there exists a subsequence u_{j_i} and $u \in L^1(\Omega)$ such that $u_{j_i} \to u$ in $L^1(\Omega)$. Furthermore, the lower semicontinuity implies that

$$\mathcal{V}(u) \le \liminf \mathcal{V}(u_{j_i}) = I.$$

It follows that $u \in \mathrm{BV}(\Omega)$. However, the trace of u need *not* be φ, and consequently u need not belong to $\mathcal{A}$. The problem here is the boundary behavior of functions in a minimizing sequence.

We must "relax" further an modify the area functional by adding the area of the piece of cylinder $\partial\Omega \times \mathbb{R}$ that lies between graphs of $\operatorname{Tr} u$ and φ. Thus we define for $u \in \mathrm{BV}(\Omega)$

$$\mathcal{J}(u) := \int_\Omega \sqrt{1+|Du|^2} + \int_{\partial\Omega} |\operatorname{Tr} u - \varphi| dH^{n-1}. \tag{7.34}$$

Theorem 7.35. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1-smooth boundary $\partial\Omega$ and let $\varphi \in L^1(\partial\Omega; H^{n-1})$. Then*

$$\inf\{\mathcal{V}(u)\colon u \in \mathrm{BV}(\Omega), \operatorname{Tr} u = \varphi\} = \inf\{\mathcal{J}(u)\colon u \in \mathrm{BV}(\Omega)\}. \tag{7.36}$$

Proof. Let us denote by L the left-side of (7.36) and by R the right-side of (7.36). Suppose that $u \in \mathrm{BV}(\Omega)$ and $\operatorname{Tr} u = \varphi$. Then $\mathcal{V}(u) = \mathcal{J}(u)$, and therefore $L \geq R$. So it remains to prove $R \geq L$. Let $u \in \mathrm{BV}(\Omega)$ and $\varepsilon > 0$. By Theorem 7.22, there exists $w \in W^{1,1}(\Omega)$, with $\operatorname{Tr} w = \varphi - \operatorname{Tr} u$ and

$$\int_\Omega |Dw| \leq (1+\varepsilon) \int_{\partial\Omega} |\operatorname{Tr} w - \varphi|\, dH^{n-1}.$$

Then $v = u + w \in \mathrm{BV}(\Omega)$ and $\operatorname{Tr} v = \varphi$. Moreover,

$$\begin{aligned}\mathcal{V}(v) &= \int_\Omega \sqrt{1+|Dv|^2} \\ &\leq \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega |Dw| \\ &\leq \int_\Omega \sqrt{1+|Du|^2} + (1+\varepsilon) \int_{\partial\Omega} |\operatorname{Tr} w - \varphi|\, dH^{n-1}.\end{aligned}$$

So, $L \leq R$. □

Problem: Given $\varphi \in L^1(\partial\Omega; H^{n-1})$ find $u \in \mathrm{BV}(\Omega)$ that minimizes

$$\mathcal{J}(v,\Omega) = \int_\Omega \sqrt{1+|Dv|^2} + \int_{\partial\Omega} |\operatorname{Tr} v - \varphi|\, dH^{n-1}$$

among $v \in \mathrm{BV}(\Omega)$. Suppose that $\bar\Omega \subset B = B^n(0,R)$. Extend φ to $\varphi \in W^{1,1}(B \setminus \bar\Omega)$ and, for each $v \in \mathrm{BV}(\Omega)$, define

$$v_\varphi = \begin{cases} v, & \text{in } \Omega; \\ \varphi, & \text{in } B \setminus \bar\Omega. \end{cases}$$

Then $v_\varphi \in \mathrm{BV}(B)$ and

$$\begin{aligned}\int_B \sqrt{1+|Dv_\varphi|^2} &= \int_\Omega \sqrt{1+|Dv|^2} + \int_{B\setminus\bar\Omega} \sqrt{1+|D\varphi|^2} + \int_{\partial\Omega} |\operatorname{Tr} v - \varphi|\, dH^{n-1} \\ &= \mathcal{J}(v,\Omega) + \int_{B\setminus\bar\Omega} \sqrt{1+|D\varphi|^2}.\end{aligned}$$

Thus we get another formulation of the problem above: Given $\varphi \in W^{1,1}(B \setminus \bar\Omega)$ find $v \in \mathrm{BV}(B)$ that minimizes

$$\mathcal{V}(v,B) = \int_B \sqrt{1+|Dv|^2}$$

among $v \in \mathrm{BV}(B)$ with $v = \varphi$ in the open set $B \setminus \bar\Omega$.

Theorem 7.37. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary $\partial\Omega$. Let $\varphi \in L^1(\partial\Omega; H^{n-1})$. Then the functional*

$$\mathcal{J}(u,\Omega) = \int_\Omega \sqrt{1+|Du|^2} + \int_{\partial\Omega} |\operatorname{Tr} u - \varphi|\, dH^{n-1}$$

attains its minimum in $\mathrm{BV}(\Omega)$.

Proof. Take an open ball B such that $\bar{\Omega} \subset B$. Extend φ to $\varphi \in W^{1,1}(B \setminus \bar{\Omega})$. Let

$$\begin{aligned} I &:= \inf \left\{ \mathcal{J}(u,\Omega) + \int_{B\setminus\bar{\Omega}} \sqrt{1+|D\varphi|^2} \colon u \in \mathrm{BV}(\Omega) \right\} \\ &= \inf \left\{ \mathcal{V}(v,B) \colon v \in \mathrm{BV}(B),\ v = \varphi \text{ in } B \setminus \bar{\Omega} \right\}. \end{aligned}$$

Since for all $v \in \mathrm{BV}(\Omega)$ the function $v_\varphi \in \mathrm{BV}(B)$ and

$$\mathcal{V}(v_\varphi, B) = \int_B \sqrt{1+|Dv_\varphi|^2} \le \int_B |Dv_\varphi|^2 + m(B) < \infty,$$

we have $I < \infty$. Take a minimizing sequence $v_j \in \mathrm{BV}(B)$, $j \in \mathbb{N}$, such that $v_j = \varphi$ in $B \setminus \bar{\Omega}$ and $\mathcal{V}(v_j, B) \to I$ as $j \to \infty$. Then v_j is a bounded sequence in $\mathrm{BV}(B)$, hence there exists a subsequence, still denoted by v_j, and $v \in \mathrm{BV}(B)$ such that $v_j \to v$ in $L^1(B)$. Clearly, $v = \varphi$ (almost everywhere) in $B \setminus \bar{\Omega}$. Hence v is "admissible" and by the lower semicontinuity,

$$\mathcal{V}(v,B) \le \liminf_{j\to\infty} \mathcal{V}(v_j, B) = I.$$

Thus $v|\Omega$ minimizes $\mathcal{J}$ in $\mathrm{BV}(\Omega)$. □

Remark 7.38. It turns out that the minimizer of $\mathcal{J}$ given by Theorem 7.37 is, in fact, smooth in Ω; see [G, Theorem 14.13].

8 The Plateau problem

In this section we study the classical Plateau problem: Given a closed curve $\Gamma \subset \mathbb{R}^3$, find a minimal surface with boundary Γ. We consider the problem for parameterized disks (solved independently by Douglas and Radó in the 30's).

Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1-smooth Jordan curve. We write $\mathbb{D} = \{z = (x,y) \in \mathbb{R}^2 \colon |z| < 1\}$ and $\mathbb{S} = \partial\mathbb{D}$ for the unit disk and for the unit circle, respectively. A map $f \colon \mathbb{S} \to \Gamma$ is called *monotone* if the preimage $f^{-1}K$ is connected for every connected $K \subset \Gamma$.

Theorem 8.1. *Given a piecewise C^1-smooth Jordan curve $\Gamma \subset \mathbb{R}^3$, there exists a map $u \colon \bar{\mathbb{D}} \to \mathbb{R}^3$ such that*

(1) $u|\mathbb{S} \colon \mathbb{S} \to \Gamma$ is monotone and onto;

(2) $u \in C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D}; \mathbb{R}^3)$ and u is C^∞-smooth in $\mathbb{D}$;

(3) the image $u(\mathbb{D})$ minimizes the area of images $v(\mathbb{D})$ among all maps $v \in C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D}; \mathbb{R}^3)$, with $v|\mathbb{S} \colon \mathbb{S} \to \Gamma$ monotone and onto.

A natural attempt to solve this problem would be to take a minimizing sequence of admissible maps and try to extract a converging subsequence. There are two serious problems in this approach. Since the area of the image is independent of parametrization, the noncompactness of the group of diffeomerphisms $\mathbb{D} \to \mathbb{D}$ is a major problem. Secondly, since thin tubes can have arbitrarily small area, the area of the image does not control the mapping enough.

8.2 Area and energy

To overcome the difficulties above, we try to minimize the energy. We denote by $W^{1,2}(\mathbb{D})$ the Sobolev space of functions $f \in L^2(\mathbb{D})$ whose distributional first order partial derivatives belong to $L^2(\mathbb{D})$ as well. It is a Banach space with the norm

$$\|f\|_{1,2} = \|f\|_p + \||\nabla f|\|_p.$$

Furthermore, we denote by $W^{1,2}(\mathbb{D};\mathbb{R}^n)$ the Banach space of mappings $f\colon \mathbb{D} \to \mathbb{R}^n$ with coordinate functions $f_i \in W^{1,2}(\mathbb{D}),\ i = 1,\dots,n$. The closure of $C_0^\infty(\mathbb{D})$ in $W^{1,2}(\mathbb{D})$ is denoted by $W_0^{1,2}(\mathbb{D})$. It is a Banach space, too.

Let $u\colon \mathbb{D} \to \mathbb{R}^n,\ u = (u^1, u^2, \dots, u^n)$, belong to $W^{1,2}(\mathbb{D};\mathbb{R}^n)$. The *energy* of u is

$$E(u) = \frac{1}{2}\int_{\mathbb{D}} |\nabla u(z)|^2\, dz,$$

where $\nabla u(z)$ $(= Du(z))$ is the linear map $\nabla u(z)\colon \mathbb{R}^2 \to \mathbb{R}^n$ defined by the distributional partial derivatives $u_x^i(z), u_y^i(z)$ as

$$\nabla u(z) = \begin{pmatrix} u_x^1(z) & u_y^1(z) \\ u_x^2(z) & u_y^2(z) \\ \vdots & \vdots \\ u_x^n(z) & u_y^n(z) \end{pmatrix}$$

and

$$|\nabla u|^2 = |u_x|^2 + |u_y|^2 = \sum_{i=1}^n \big((u_x^i)^2 + (u_y^i)^2\big)$$

is the square of the Hilbert-Schmidt norm of ∇u. Notice that $\nabla u(z)$ is defined at almost every point $z = (x, y) \in \mathbb{D}$. We define the area of the image $u(\mathbb{D})$ (with multiplicity counted) by

$$\mathrm{Area}(u) = \int_{\mathbb{D}} \big(|u_x|^2|u_y|^2 - \langle u_x, u_y\rangle^2\big)^{1/2}.$$

Note that the integrand

$$|u_x \wedge u_y| = \big(|u_x|^2|u_y|^2 - \langle u_x, u_y\rangle^2\big)^{1/2}$$

is the area of the parallelogram spanned by vectors $u_x = (\nabla u)e_1,\ u_y = (\nabla u)e_2 \in \mathbb{R}^n$. Hence $\mathrm{Area}(u)$ is the 2-dimensional (normalized) Hausdoff measure of the image (at least) if u is C^1 and one-to-one. Since

$$E(u) = \frac{1}{2}\int_{\mathbb{D}} \big(|u_x|^2 + |u_y|^2\big)$$

we notice that $\mathrm{Area}(u) \le E(u)$ and the equality occurs if and only if

$$\langle u_x, u_y\rangle = 0 \text{ and } |u_x| = |u_y| \text{ a.e.} \tag{8.3}$$

That is $\langle u_x, u_y\rangle = 0$ and $|u_x|^2 - |u_y|^2 = 0$ as L^1-functions. We call a mapping $u \in W^{1,2}(\mathbb{D};\mathbb{R}^n)$ *almost conformal* if $\mathrm{Area}(u) = E(u)$. Thus (8.3) holds a.e., but $\nabla u(z)$ may be 0 (constant linear map) in a set of positive measure. If, furthermore, u is an almost conformal immersion, then u is conformal, hence an isothermal parametrization. The existence of isothermal coordinates in 2-dimensions will be crucial for the proof and the reason why this kind of approach does not work for higher dimensional analogues. Indeed, provided certain properties (see below) hold for a map

$u \in W^{1,2}(\mathbb{D};\mathbb{R}^n)$, there exists a homeomorphism $\varphi\colon \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ such that $\varphi|\mathbb{D}\colon \mathbb{D} \to \mathbb{D}$ is diffeomorphic and $u \circ \varphi$ is conformal.

Fix a piecewise C^1-smooth Jordan curve $\Gamma \subset \mathbb{R}^3$ and denote by X_Γ the family of admissible maps

$$X_\Gamma = \{\psi\colon \bar{\mathbb{D}} \to \mathbb{R}^3 \mid \psi \in C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D},\mathbb{R}^3),\ \psi|\mathbb{S}\colon \mathbb{S} \to \Gamma \text{ monotone}\}.$$

Furthermore, let

$$\begin{aligned} A_\Gamma &= \inf\{\mathrm{Area}(u)\colon u \in X_\Gamma\} \\ E_\Gamma &= \inf\{E(u)\colon u \in X_\Gamma\}. \end{aligned}$$

It is easy to see that $X_\Gamma \neq \emptyset$, and therefore the definitions of A_Γ and E_Γ make sense.

Lemma 8.4. $A_\Gamma = E_\Gamma$.

Proof. Since $\mathrm{Area}(u) \leq E(u)$, we have $A_\Gamma \leq E_\Gamma$. The proof of the converse inequality relies on the existence of isothermal coordinates. Let us fix $\varepsilon > 0$ and choose $u \in X_\Gamma$ such that

$$\mathrm{Area}(u) < A_\Gamma + \varepsilon/2.$$

Recall that

$$\mathrm{Area}(u) = \int_{\mathbb{D}} \left(|u_x|^2|u_y|^2 - \langle u_x, u_y\rangle^2\right)^{1/2} dx\, dy.$$

Next we pull-back the standard inner product (Riemannian metric) of $\mathbb{R}^3$ by the mapping u to obtain a measurable, possibly degenerate, "Riemannian metric" on $\mathbb{D}$. Writing $u = (u^1, u^2, u^3)$ and

$$\begin{aligned} g_{11} &= \langle u_x, u_x\rangle = |u_x|^2, \\ g_{22} &= \langle u_y, u_y\rangle = |u_y|^2, \\ g_{12} &= g_{21} = \langle u_x, u_y\rangle, \end{aligned}$$

and

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} |u_x|^2 & \langle u_x, u_y\rangle \\ \langle u_x, u_y\rangle & |u_y|^2 \end{pmatrix}, \tag{8.5}$$

we see that

$$\left(|u_x|^2|u_y|^2 - \langle u_x, u_y\rangle^2\right)^{1/2} = \sqrt{\det g_{ij}}$$

and

$$|\nabla u|^2 = |u_x|^2 + |\nabla u_y|^2 = \mathrm{tr}\, g_{ij}.$$

Indeed, g_{ij} is the pull-back of $\langle\cdot,\cdot\rangle$ under u and it defines, for a.e. $z \in \mathbb{D}$, a symmetric bilinear form $\langle\cdot,\cdot\rangle_z\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\langle e_i, e_j\rangle_z = \nabla u(z)e_i \cdot \nabla u(z)e_j = u_x(z) \cdot u_y(z) = g_{ij}(z).$$

Notice that $\langle\cdot,\cdot\rangle_z$ fails to be an inner product at points $z \in \mathbb{D}$, where $\nabla u(z)$ is not injective. Our aim is to change the "metric" g_{ij} into a conformal one ($= f\delta_{ij}$, $f > 0$ a function). In order to use the existence results for isothermal coordinates g_{ij} should be non-degenerate. To solve this we problem we first approximate u by a sequence of smooth mappings. Indeed, since $u \in C(\bar{\mathbb{D}};\mathbb{R}^3)$ there exists a sequence $u_{(k)} \in C^\infty(\bar{\mathbb{D}};\mathbb{R}^3)$ such that $u_{(k)} \to u$ uniformly in $\bar{\mathbb{D}}$ and the partial derivatives of coordinate functions converge in $L^2(\mathbb{D})$, i.e. $u^i_{(k)x} \to u^i_x$ in $L^2(\mathbb{D})$ and $u^i_{(k)y} \to u^i_y$ in $L^2(\mathbb{D})$ as

$k \to \infty$ $(i = 1, 2, 3)$. We denote by $g_{ij}^{(k)}$ the pull-back of $\langle \cdot, \cdot \rangle$ under $u_{(k)}$ (see (8.5)). For each k we then define a family of mappings $u_{(k,s)} \in C^\infty(\bar{\mathbb{D}}; \mathbb{R}^5)$, $s \in \mathbb{R}$, by

$$u_{(k,s)}(x, y) = \big(\underbrace{u_{(k)}(x, y)}_{\in \mathbb{R}^3}, sx, sy\big).$$

Then

$$\nabla u_{(k,s)} = \begin{pmatrix} u^1_{(k)x} & u^1_{(k)y} \\ u^2_{(k)x} & u^2_{(k)y} \\ u^3_{(k)x} & u^3_{(k)y} \\ s & 0 \\ 0 & s \end{pmatrix}$$

and the pull-back of the standard inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^5$ under $u_{(k,s)}$ is $\tilde{g}_{ij} = g_{ij}^{(k)} + s^2 \delta_{ij}$. Furthermore,

$$\det \tilde{g}_{ij} = \det g_{ij}^{(k)} + s^2 |\nabla u_{(k)}|^2 + s^4,$$

and therefore $\tilde{g}_{ij}$ is non-degenerate and smooth for $s \neq 0$. We also notice that

$$\sqrt{\det \tilde{g}_{ij}} \leq \sqrt{\det g_{ij}^{(k)}} + |s||\nabla u_{(k)}| + s^2.$$

The existence results for isothermal coordinates imply that, for $s \neq 0$ and $k \in \mathbb{N}$, there exists a homeomorphism $\varphi_{k,s} \colon \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ such that $\varphi_{k,s}|\mathbb{D} \colon \mathbb{D} \to \mathbb{D}$ is diffeomorphic and

$$u_{(k,s)} \circ \varphi_{k,s} \colon \mathbb{D} \to \mathbb{R}^5$$

is conformal. Since

$$\nabla \left(u_{(k,s)} \circ \varphi_{k,s}\right) = \begin{pmatrix} \big(u^1_{(k)} \circ \varphi_{k,s}\big)_x & \big(u^1_{(k)} \circ \varphi_{k,s}\big)_y \\ \big(u^2_{(k)} \circ \varphi_{k,s}\big)_x & \big(u^2_{(k)} \circ \varphi_{k,s}\big)_y \\ \big(u^3_{(k)} \circ \varphi_{k,s}\big)_x & \big(u^3_{(k)} \circ \varphi_{k,s}\big)_y \\ s\big(\varphi^1_{k,s}\big)_x & 0 \\ 0 & s\big(\varphi^2_{k,s}\big)_y \end{pmatrix},$$

we have

$$\begin{aligned} \left|\nabla \left(u_{(k,s)} \circ \varphi_{k,s}\right)\right|^2 &= \left|\nabla \left(u_{(k)} \circ \varphi_{k,s}\right)\right|^2 + s^2 \big(\varphi^1_{k,s}\big)_x^2 + s^2 \big(\varphi^2_{k,s}\big)_y^2 \\ &\geq \left|\nabla \left(u_{(k)} \circ \varphi_{k,s}\right)\right|^2. \end{aligned}$$

Hence

$$\begin{aligned}
E\big(u_{(k)}\circ\varphi_{k,s}\big) &= \frac{1}{2}\int_{\mathbb{D}}\big|\nabla\big(u_{(k)}\circ\varphi_{k,s}\big)\big|^2\\
&\le \frac{1}{2}\int_{\mathbb{D}}\big|\nabla\big(u_{(k,s)}\circ\varphi_{k,s}\big)\big|^2\\
&= E\big(u_{(k,s)}\circ\varphi_{k,s}\big)\\
&= \mathrm{Area}\big(u_{(k,s)}\circ\varphi_{k,s}\big)\\
&= \mathrm{Area}\big(u_{(k,s)}\big),
\end{aligned}$$

where the second last equality follows from conformality of $u_{(k,s)}\circ\varphi_{k,s}$ and the last equality follows since the area is independent of parametrization. On the other hand,

$$\begin{aligned}
\mathrm{Area}\big(u_{(k,s)}\big) &= \int_{\mathbb{D}}\sqrt{\det\tilde{g}_{ij}}\\
&\le \int_{\mathbb{D}}\left(\sqrt{\det g_{ij}^{(k)}}+|s||\nabla u_{(k)}|+s^2\right)\\
&= \int_{\mathbb{D}}\sqrt{\det g_{ij}^{(k)}}+|s|\int_{\mathbb{D}}|\nabla u_{(k)}|+\pi s^2.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\mathbb{D}}|\nabla u_{(k)}| &= \int_{\mathbb{D}}|\nabla u_{(k)}-\nabla u+\nabla u|\\
&\le \int_{\mathbb{D}}|\nabla u_{(k)}-\nabla u|+\int_{\mathbb{D}}|\nabla u|\\
&\le \sqrt{\pi}\underbrace{\left(\int_{\mathbb{D}}|\nabla u_{(k)}-\nabla u|^2\right)^{1/2}}_{\to 0}+\sqrt{\pi}\underbrace{\left(\int_{\mathbb{D}}|\nabla u|^2\right)^{1/2}}_{<\infty},
\end{aligned}$$

we may find s and $k_0\in\mathbb{N}$ such that

$$|s|\int_{\mathbb{D}}|\nabla u_{(k)}|+\pi s^2<\varepsilon/4$$

for all $k\ge k_0$. On the other hand, since $u_{(k)}\to u$ in $W^{1,2}(\mathbb{D},\mathbb{R}^3)$ we may choose k_0 so large that

$$\int_{\mathbb{D}}\sqrt{\det g_{ij}^{(k)}}\le\int_{\mathbb{D}}\sqrt{\det g_{ij}}+\varepsilon/4=\mathrm{Area}(u)+\varepsilon/4,$$

and consequently,

$$E\big(u_{(k)}\circ\varphi_{k,s}\big)\le\mathrm{Area}\big(u_{(k,s)}\big)\le\mathrm{Area}(u)+\varepsilon/2<A_\Gamma+\varepsilon$$

for all $k\ge k_0$. It follows that

$$E_\Gamma\le E(u)\le\liminf_{k\to\infty}E(u_{(k)})=\liminf_{k\to\infty}E\big(u_{(k)}\circ\varphi_{k,s}\big)\le A_\Gamma+\varepsilon.$$

□

Remark 8.6. In this remark we explain the existence of isothermal coordinates in the setting of Lemma 8.4. Using classical notation we write the metric $\tilde{g}_{ij}$, i.e. the pull-back of the standard inner product $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^5$ under a smooth map $u_{k,s}$, as

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2,$$

where

$$\begin{aligned} E &= |u_{(k)x}|^2 + s^2, \\ G &= |u_{(k)y}|^2 + s^2, \quad \text{and} \\ F &= \langle u_{(k)x}, u_{(k)y}\rangle \end{aligned}$$

are smooth in $\bar{\mathbb{D}}$. In complex coordinates $z = x + iy$, the metric $\tilde{g}_{ij}$ reads as

$$ds^2 = \lambda|dz + \mu d\bar{z}|^2,$$

where

$$\lambda = \frac{1}{4}(E + G + 2\sqrt{EG - F^2}) > 0$$

and

$$\mu = \frac{1}{4\lambda}(E - G + 2iF)$$

are smooth in $\bar{\mathbb{D}}$ and $\|\mu\|_\infty < 1$. In isothermal coordinates (ζ_1, ζ_2) a metric should take a form

$$ds^2 = \rho(d\zeta_1^2 + d\zeta_2^2) \tag{8.7}$$

which using complex notation $\omega = \zeta_1 + i\zeta_2$ reads as

$$ds^2 = \rho|d\omega|^2 = \rho|\omega_z dz + \omega_{\bar{z}} d\bar{z}|^2 = \rho|\omega_z|^2\Big|dz + \frac{\omega_{\bar{x}}}{\omega_z}d\bar{z}\Big|^2,$$

with $\rho > 0$ smooth. The idea in the proof of Lemma 8.4 is to compose $u_{k,s}$ with a diffeomorphism $\varphi_{k,s}\colon \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ in such a way that the pull-back of $\langle\cdot,\cdot\rangle$ under the map $u_{(k,s)} \circ \varphi_{k,s}$ is of the form (8.7). Such a map $\varphi_{k,s}$ can be found as follows. First we extend μ to (a complex dilatation) $\mu \in C_0^\infty(\mathbb{C})$, with $\|\mu\|_\infty < 1$. Then the principal solution $f\colon \mathbb{C} \to \mathbb{C}$ to the *Beltrami equation*

$$f_{\bar{z}} = \mu f_z$$

is a C^∞-smooth quasiconformal diffeomorphism[1]. Next we apply the Riemann mapping theorem to find a homeomorphism $\psi\colon \overline{f(\mathbb{D})} \to \bar{\mathbb{D}}$ such that $\psi|\mathbb{D}$ is conformal. Then $f^{-1} \circ \psi^{-1}$ is a mapping $\varphi_{k,s}$ we are looking for.

Theorem 8.1 will be proven in two steps. First we show that for each parametrization $\mathbb{S} \to \Gamma$ of the boundary Jordan curve Γ, there exists an energy minimizing continuous map $u\colon \bar{\mathbb{D}} \to \mathbb{R}^3$, $u(\mathbb{S}) = \Gamma$. Each such minimizer u is a harmonic map which in the case of the Euclidean target $\mathbb{R}^3$ means that coordinate functions u^1, u^2, u^3 are harmonic functions. By Weyl's lemma u^i's and hence u will be C^∞. This part follows from the classical Dirichlet problem for harmonic functions. Secondly, we minimize the energies over possible parametrizations of Γ. In this step the difficulty is in extracting a convergent subsequence. That problem will be solved by using the Courant-Lebesgue lemma.

[1]See e.g. Section 5.2 in Astala-Iwaniec-Martin: Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton University Press, 2009.

8.8 Dirichlet problem

Next we will solve the following Dirichlet problem.

Theorem 8.9. *Given $h \in C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$, there exists a unique harmonic function $u \in C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$, with $u|\partial\mathbb{D} = h|\partial\mathbb{D}$.*

Although we could rely on the results in Section 6, we will present a more specific approach. First we give some preliminary results.

Preliminaries

Since $W^{1,2}(\mathbb{D})$ is a reflexive Banach space, we have the following weak compactness:

Lemma 8.10. *If $u_k \in W^{1,2}(\mathbb{D})$ is a sequence such that*

$$\sup_k \|u_k\|_{1,2} < \infty,$$

then there exists a subsequence u_{k_j} and $u \in W^{1,2}(\mathbb{D})$ such that $u_{k_j} \to u$ weakly in $L^2(\mathbb{D})$, $\nabla u_{k_j} \to \nabla u$ weakly in $L^2(\mathbb{D};\mathbb{R}^2)$, and

$$\int_{\mathbb{D}} |\nabla u|^2 \leq \liminf \int_{\mathbb{D}} |\nabla u_{k_j}|^2.$$

Moreover, if each $u_k \in W_0^{1,2}(\mathbb{D})$, then $u \in W_0^{1,2}(\mathbb{D})$.

Another ingredient will be the following Poincaré inequality.

Lemma 8.11. *There exists a constant C such that*

$$\int_{\mathbb{D}} |u|^2 \, dx \leq C \int_{\mathbb{D}} |\nabla u|^2 \, dx \tag{8.12}$$

for every $u \in W_0^{1,2}(\mathbb{D})$.

Proof. We will sketch a proof in a more general setting. Suppose that $D \subset \mathbb{R}^n, n \geq 2$, is a bounded open set. Then there exists a constant $C = C(D) < \infty$ such that

$$\int_D u \, dm \leq C \int_D |\nabla u| \, dm$$

for every nonnegative $u \in C_0^\infty(D)$. The Poincaré inequality follows from this by Hölder's inequality and approximation. Let $R = \operatorname{diam}(\bar{D})$. Without loss of generality we may assume that $D \subset \bar{B}(0, R+\varepsilon) \setminus B(0,\varepsilon)$. Let $r(x) = |x|$. Then

$$\Delta r(x) = \frac{n-1}{r(x)}$$

in $\mathbb{R}^n \setminus \{0\}$. In particular, in D we have

$$\frac{n-1}{R+\varepsilon} \leq \Delta r(x) \leq \frac{n-1}{\varepsilon},$$

and therefore by integration by parts

$$\frac{n-1}{R+\varepsilon} \int_D u \, dm \leq \int_D u \Delta r \, dm = -\int_G \nabla u \cdot \nabla r \, dm \leq \int_D |\nabla| \underbrace{|\nabla r|}_{=1} \, dm = \int_D |u| \, dm.$$

□

We say that a function $u \in L^1_{\text{loc}}(\mathbb{D})$ is *weakly harmonic* if

$$\int_{\mathbb{D}} u\Delta\varphi = 0 \tag{8.13}$$

for every $\varphi \in C_0^\infty(\mathbb{D})$. If $u \in W^{1,2}_{\text{loc}}(\mathbb{D})$, this is equivalent to

$$\int_D \langle \nabla u, \nabla\varphi\rangle = 0$$

for every $\varphi \in C_0^\infty(\mathbb{D})$.

Lemma 8.14 (Weyl's lemma)**.** *If $u \in L^1_{\text{loc}}(\mathbb{D})$ is weakly harmonic, then $u \in C^\infty(\mathbb{D})$.*

Proof. The proof is based on the mean value property of weakly harmonic functions (see Exercises 10/1-10/5). More precisely, if $u \in L^1_{\text{loc}}(\mathbb{D})$ is weakly harmonic, then there exists $\bar{u} \in C(\mathbb{D})$ such that $u = \bar{u}$ a.e. in $\mathbb{D}$ and that

$$2\pi\bar{u}(y) = \int_0^{2\pi} \bar{u}\big(y + (r\cos\theta, r\sin\theta)\big)\, d\theta$$

for every $y \in \mathbb{D}$ and $r > 0$, with $B^2(y,r) \in \mathbb{D}$. We will next show that $\bar{u}$ is smooth. We identify u and $\bar{u}$ as elements in $L^1(\mathbb{D})$ and thus we write $u = \bar{u}$. Fix a C^∞, nonnegative and nonincreasing function $\psi\colon [0,\infty) \to [0,\infty)$ such that

$$\psi|[0,1/3] = \text{const.}, \quad \operatorname{supp}\psi \subset [0,2/3],$$

and

$$2\pi \int_0^1 \psi(t)\, dt = 1.$$

For $t \in (0,1)$ define a C^∞-smooth function $\varphi_t\colon \mathbb{R}^2 \to [0,\infty)$ by setting

$$\varphi_t(x) = t^{-2}\psi(|x|/t).$$

Then φ_t is radially symmetric, $\operatorname{supp}\varphi \subset B^2(0,t)$, and

$$\int_{\mathbb{R}^2} \varphi_t = 1.$$

Fix $t \in (0,1)$, and let $u_t = u * \varphi_t\colon B^2(0,1-t) \to \mathbb{R}$ be the convolution of u and φ_t,

$$u_t(y) = \int_{\mathbb{R}^2} u(y+x)\varphi_t(x)\, dx = \int_{\mathbb{R}^2} u(z)\varphi_t(z-y)\, dz, \quad |y| < 1-t.$$

Then u_t is C^∞-smooth. We will show that $u_t = u$ which proves the claim. Let $y \in \mathbb{D}$ and suppose that $y \in B^2(0,1-t)$ for some $t \in (0,1)$. Using polar coordinates, we have

$$\begin{aligned}
u_t(y) &= \int_0^t \int_0^{2\pi} u\big(\underbrace{y + (r\cos\theta, r\sin\theta)}_{=y+x}\big)\varphi_t(\underbrace{\cos\theta, r\sin\theta}_{=x})r\, drd\theta \\
&= \int_0^t t^{-2}\psi(r/|x|) \underbrace{\int_0^{2\pi} u\big(y + (r\cos\theta, r\sin\theta)\big)\, d\theta}_{=2\pi u(y)^t}\, r\, dr \\
&= 2\pi u(y) \int_0 t^{-2}\psi(r/|x|) r\, dr \\
&= u(y) \int_0^{2\pi} \int_0 t^{-2}\psi(r/|x|) r\, drd\theta \\
&= u(y).
\end{aligned}$$

□

Boundary regularity

Lemma 8.15. *Suppose that $h \in C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$. If u is weakly harmonic and $u - h \in W_0^{1,2}(\mathbb{D})$, then u has a continuous extension $u \in C(\bar{\mathbb{D}})$ such that $u|\partial\mathbb{D} = h|\partial\mathbb{D}$.*

Proof. By Weyl's lemma $u \in C^\infty(\mathbb{D})$, so it remains to prove that

$$\lim_{x \to y} u(x) = h(y) \quad \forall y \in \partial\mathbb{D}.$$

We may assume without loss of generality that $y = (1,0) \in \partial\mathbb{D}$. The function

$$x = (x_1, x_2) \mapsto 1 - x_1$$

is harmonic and continuous in $\mathbb{R}^2$, positive in $\bar{\mathbb{D}} \setminus \{y\}$, and vanishes at y. Fix $\varepsilon > 0$. By continuity of h, there exists $\delta > 0$ such that

$$|h(z) - h(y)| < \varepsilon \quad \forall z \in \bar{\mathbb{D}} \cap B^2(y, \delta).$$

Since $1 - x_1 > 0$ in $\bar{\mathbb{D}} \setminus B^2(y,\delta)$, there exists $k > 0$ such that

$$k(1 - x_1) > 2 \sup_{\mathbb{D}} |h| \quad \forall (x_1, x_2) \in \bar{\mathbb{D}} \setminus B^2(y, \delta).$$

Define functions h^+ and h^- by

$$h^+(x_1, x_2) = h(y) + \big(\varepsilon + k(1 - x_1)\big),$$
$$h^-(x_1, x_2) = h(y) - \big(\varepsilon + k(1 - x_1)\big).$$

Then h^+ and h^- are harmonic and

$$h^- \le h \le h^+$$

in $\bar{\mathbb{D}}$, in particular, on $\partial\mathbb{D}$. Since $u - h \in W_0^{1,2}(\mathbb{D})$, it follows form the maximum principle (Exerc.) that

$$h^- \le u \le h^+$$

in $\bar{\mathbb{D}}$. Since $(x_1, x_2) \mapsto 1 - x_1$ is continuous and vanishes at $y = (1,0)$, we can choose $\delta_0 > 0$ so small that $k(1 - x_1) < \varepsilon$ in $\bar{\mathbb{D}} \cap B^2(y, \delta_0)$. Then

$$|u - h(y)| < 2\varepsilon$$

in $\mathbb{D} \cap B^2(y, \delta_0)$. □

Solving the Dirichlet problem

We will use direct methods in calculus of variations. Let $h \in W^{1,2}(\mathbb{D})$ and denote

$$\mathcal{A}_h = \{v \in W^{1,2}(\mathbb{D})\colon v - h \in W_0^{1,2}(\mathbb{D})\}$$

and

$$E_h = \inf_{v \in \mathcal{A}_h} \int_{\mathbb{D}} |\nabla v|^2.$$

Since $h \in \mathcal{A}_h$ we notice that $E_h < \infty$. Choose a minimizing sequence $u_j \in \mathcal{A}_h$ so that

$$E_h \leq \int_{\mathbb{D}} |\nabla u_j|^2 < E_h + 1/j.$$

Recall the parallelogram law in an inner product space:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Thus we have

$$\left|\nabla\left(\frac{u_i + u_j}{2}\right)\right|^2 + \left|\nabla\left(\frac{u_i - u_j}{2}\right)\right|^2 = 2\left(\left|\frac{\nabla u_i}{2}\right|^2 + \left|\frac{\nabla u_j}{2}\right|^2\right) = \frac{1}{2}|\nabla u_i|^2 + \frac{1}{2}|\nabla u_j|^2.$$

Integrating over $\mathbb{D}$ gives

$$\frac{1}{4}\int_{\mathbb{D}} |\nabla(u_i - u_j)|^2 + \int_{\mathbb{D}} \left|\nabla\left(\frac{u_i + u_j}{2}\right)\right|^2 = \frac{1}{2}\int_{\mathbb{D}} |\nabla u_i|^2 + \frac{1}{2}\int_{\mathbb{D}} |\nabla u_j|^2 < E_h + \frac{1}{2}(1/i + 1/j).$$

Since $(u_i + u_j)/2 \in \mathcal{A}_h$, we have

$$\int_{\mathbb{D}} \left|\nabla\left(\frac{u_i + u_j}{2}\right)\right|^2 \geq E_h,$$

and therefore

$$\frac{1}{4}\int_{\mathbb{D}} |\nabla(u_i - u_j)|^2 < \frac{1}{2}(1/i + 1/j).$$

On the other hand, $u_i - u_j \in W_0^{1,2}(\mathbb{D})$, and hence

$$\int_{\mathbb{D}} |u_i - u_j|^2 \leq C\int_{\mathbb{D}} |\nabla(u_i - u_j)|^2 < 2C(1/i + 1/j)$$

by the Poincaré inequality. It follows that (u_i) is a Cauchy sequence in $W^{1,2}(\mathbb{D})$ and since $W^{1,2}(\mathbb{D})$ is a Banach space, there exists $u \in W^{1,2}(\mathbb{D})$ such that $u_i \to u$ in $W^{1,2}(\mathbb{D})$. By lower semicontinuity,

$$\int_{\mathbb{D}} |\nabla u|^2 \leq \liminf \int_{\mathbb{D}} |\nabla u_i|^2 = E_h.$$

Furthermore, since $u_i - h \in W_0^{1,2}(\mathbb{D})$, also $u - h \in W_0^{1,2}(\mathbb{D})$, and therefore $u \in \mathcal{A}_h$. Thus u is a minimizer and hence (weakly) harmonic in $\mathbb{D}$.

We have thus proved Theorem 8.9.

8.16 Controlling boundary parametrizations

Lemma 8.17 (Courant-Lebesgue lemma). *Suppose that $u\colon \mathbb{D} \to \mathbb{R}^3$ is a mapping that belongs to $C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D}; \mathbb{R}^3)$ with energy $E(u) < K/2$ for some $K > 0$. For $y \in \bar{\mathbb{D}}$ and $r > 0$ denote*

$$\begin{aligned} C_r &= \{x \in \bar{\mathbb{D}} \colon |x - y| = r\}, \\ d(C_r) &= \operatorname{diam}\big(u(C_r)\big), \\ L(C_r) &= \textit{ length of } u(C_r) = H^1\big(u(C_r)\big). \end{aligned}$$

Then for every $\delta \in (0,1)$, *there exists* $\varrho \in [\delta, \sqrt{\delta}]$ *such that*

$$\big(d(C_\varrho)\big)^2 \le 2\pi\varepsilon_\delta,$$

where

$$\varepsilon_\delta = \frac{4\pi K}{\log(1/\delta)}.$$

Proof. By approximation we may assume that $u \in C^1(\mathbb{D}) \cap W^{1,2}(\mathbb{D})$ (Exerc.). Define

$$p(r) = r \int_{C_r} |\nabla u|^2\, ds$$

where the integration is with respect to arc-length. Then

$$\begin{aligned}
\int_\delta^{\sqrt{\delta}} p(r)\, d(\log r) &= \int_\delta^{\sqrt{\delta}} p(r)\, \tfrac{dr}{r} \\
&= \int_\delta^{\sqrt{\delta}} \int_{C_r} |\nabla u|^2\, ds dr \\
&= \int_{\mathbb{D}} |\nabla u|^2 \le K.
\end{aligned}$$

The mean value theorem implies that there exists $\varrho \in [\delta, \sqrt{\delta}]$ such that

$$\begin{aligned}
\int_\delta^{\sqrt{\delta}} p(r)\, d(\log r) &= p(\varrho) \int_\delta^{\sqrt{\delta}} d(\log r) \\
&= p(\varrho) \log \frac{\sqrt{\delta}}{\delta} \\
&= \frac{1}{2} p(\varrho) \log(1/\delta).
\end{aligned}$$

Hence

$$p(\varrho) \le \frac{2K}{\log(1/\delta)}.$$

For every $r \in [\delta, \sqrt{\delta}]$,

$$L(C_r) = \int_{u(C_r)} ds \le \int_{C_r} |\nabla u|\, ds \le \left(\int_{C_r} |\nabla u|^2\, ds\right)^{1/2} (2\pi r)^{1/2},$$

and therefore

$$(L(C_r))^2 \le 2\pi r \int_{C_r} |\nabla u|^2\, ds = 2\pi p(r).$$

In particular,

$$\big(d(C_\varrho)\big)^2 \le (L(C_\varrho))^2 \le 2\pi p(\varrho) \le 4\pi K/\log(1/\delta).$$

$\square$

Next we prove that in dimension 2 the (2-)energy is invariant under conformal diffeomorphisms. We just need the following special case:

Lemma 8.18. *Let $u \in W^{1,2}(\mathbb{D})$ and let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a conformal diffeomorphism. Then*
$$E(u) = E(u \circ \varphi).$$

Proof. Recall that the differential and the gradient of a function are related by
$$du(V) = \langle \nabla u, V \rangle$$
for every vector field V. By the chain rule (which holds also for Sobolev functions),
$$d(u \circ \varphi) = du \circ D\varphi,$$
hence
$$\begin{aligned}\langle \nabla(u \circ \varphi)(z), V_z \rangle &= d(u \circ \varphi)_z(V_z) = du_{\varphi(z)}(D\varphi_z V_z)\\ &= \langle \nabla u\big(\varphi(z)\big), D\varphi_z V_z \rangle = \langle D\varphi^*_{\varphi(z)} \nabla u\big(\varphi(z)\big), V_z \rangle\end{aligned}$$
for every V. We obtain
$$\nabla(u \circ \varphi)(z) = D\varphi^*_{\varphi(z)} \nabla u\big(\varphi(z)\big).$$
Since $\varphi = (\varphi^1, \varphi^2)$ is conformal, it satisfies the Cauchy-Riemann equations
$$\begin{cases}\varphi^1_x = \varphi^2_y\\ \varphi^2_x = -\varphi^2_x.\end{cases}$$
Hence the matrices of $D\varphi$ and $D\varphi^*$ are
$$D\varphi = \begin{pmatrix}\varphi^1_x & \varphi^1_y\\ \varphi^2_x & \varphi^2_y\end{pmatrix} = \begin{pmatrix}\varphi^1_x & -\varphi^2_x\\ \varphi^2_x & \varphi^1_x\end{pmatrix}$$
and
$$D\varphi^* = \begin{pmatrix}\varphi^1_x & \varphi^2_x\\ -\varphi^2_x & \varphi^1_x\end{pmatrix}.$$
Thus the Jacobian determinant of φ is
$$J_\varphi = \det D\varphi = \varphi^1_x\varphi^2_y - \varphi^1_y\varphi^2_x = (\varphi^1_x)^2 + (\varphi^2_x)^2.$$
Since
$$D\varphi^*\nabla u = \begin{pmatrix}\varphi^1_x & \varphi^2_x\\ -\varphi^2_x & \varphi^1_x\end{pmatrix}\begin{pmatrix}u_x\\ u_y\end{pmatrix} = \begin{pmatrix}\varphi^1_x u_x + \varphi^2_x u_y\\ -\varphi^2_x u_x + \varphi^1_x u_y\end{pmatrix},$$
we have
$$\begin{aligned}|D\varphi^*\nabla u|^2 &= (\varphi^1_x u_x + \varphi^2_x u_y)^2 + (-\varphi^2_x u_x + \varphi^1_x u_y)^2\\ &= (\varphi^1_x)^2|\nabla u|^2 + (\varphi^2_x)^2|\nabla u|^2\\ &= J_\varphi|\nabla u|^2.\end{aligned}$$
Hence
$$\begin{aligned}E(u \circ \varphi) &= \frac{1}{2}\int_{\mathbb{D}} |\nabla(u \circ \varphi)(z)|^2\,dz\\ &= \frac{1}{2}\int_{\mathbb{D}} \left|D\varphi^*_{\varphi(z)}\nabla u\big(\varphi(z)\big)\right|^2\,dz\\ &= \frac{1}{2}\int_{\mathbb{D}} \big|\nabla u\big(\varphi(z)\big)\big|^2 J_\varphi(z)\,dz\\ &= \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2\\ &= E(u).\end{aligned}$$

□

Recall from complex analysis that the group of Möbius transformations $\varphi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ acts triply-transitively, that is, given two triples (p_1, p_2, p_3) and (q_1, q_2, q_3) of distinct points in $\hat{\mathbb{C}}$, there exists a unique Möbius transformation φ such that $\varphi(p_i) = q_i,\ i = 1, 2, 3$. Moreover, if the triples (p_1, p_2, p_3) and (q_1, q_2, q_3) have the same orientation with respect to $\mathbb{D}$, then the map φ maps $\bar{\mathbb{D}}$ to $\bar{\mathbb{D}}$.

Let then $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1 Jordan curve and fix a triple (p_1, p_2, p_3) on $\partial\mathbb{D}$ and a triple (q_1, q_2, q_3) on Γ. We have proved:

Lemma 8.19. *Let $u\colon \bar{\mathbb{D}} \to \mathbb{R}^3$ be a mapping belonging to $C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$ such $u|\partial\mathbb{D}\colon \partial\mathbb{D} \to \Gamma$ is monotone and onto. Then there exists a Möbius transformation $\varphi\colon \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ such that*

(1) $E(u \circ \varphi) = E(u)$;

(2) $u \circ \varphi \in C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$;

(3) $u \circ \varphi|\partial\mathbb{D}\colon \partial\mathbb{D} \to \Gamma$ is monotone and onto;

(4) $\{u(\varphi(p_1)), u(\varphi(p_2)), u(\varphi(p_3))\} = \{q_1, q_2, q_3\}$.

We will use the Courant-Lebesgue lemma 8.17 to prove that fixing the images of three points and bounding the energy implies equicontinuity on $\partial\mathbb{D}$.

Lemma 8.20. *For $K > 0$, let $\mathcal{F}_K$ be the family of all maps $\psi\colon \bar{\mathbb{D}} \to \mathbb{R}^3$ satisfying*

(1) $\psi \in C(\bar{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$ and $E(\psi) \le K/2$;

(2) $\psi|\partial\mathbb{D}\colon \partial\mathbb{D} \to \Gamma$ is monotone and onto;

(3) $\psi(\{p_1, p_2, p_3\}) = \{q_1, q_2, q_3\}$.

Then $\mathcal{F}_K$ is equicontinuous on $\partial\mathbb{D}$.

Proof. Fix $\varepsilon > 0$ smaller than $\min\{|q_i - q_j|\colon i \ne j\}$. Since Γ is a piecewise C^1 Jordan curve, there exists $d > 0$ such that $\Gamma \setminus \{p, q\}$ has exactly one component of diameter $\le \varepsilon$ whenever $p, q \in \Gamma$ are points with $0 < |p - q| < d$. Fix $\delta \in (0, 1)$ such that $\sqrt{2\pi\varepsilon_\delta} < d$ and that at most one of the points p_i belong to $B^2(z, \sqrt{\delta})$ for every $z \in \partial\mathbb{D}$. By the Courant-Lebesgue lemma 8.17, given any $z \in \partial\mathbb{D}$, there exists $\varrho_\psi \in [\delta, \sqrt{\delta}]$ such that

$$d(C_{\varrho_\psi}) \le \sqrt{2\pi\varepsilon_\delta} < d,$$

where $d(C_{\varrho_\psi})$ is the diameter of the image $\psi(C_{\varrho_\psi})$ of the circular arc $C_{\varrho_\psi} = \{y \in \bar{\mathbb{D}}\colon |y - z| = \varrho_\psi\}$ and

$$\varepsilon_\delta = \frac{4\pi K}{\log(1/\delta)}.$$

The circular arc C_{ϱ_ψ} divides the unit circle $\partial\mathbb{D}$ into two components $A_{1,\psi}$ and $A_{2,\psi}$ such that the longer one, say $A_{2,\psi}$, contains at least two of the points p_i. Denote by $\mathcal{A}_{i,\psi} = \psi A_{i,\psi} \subset \Gamma$ the corresponding images. If $\{\zeta_1, \zeta_2\} \subset \partial\mathbb{D} \cap C_{\varrho_\psi}$ are the endpoints of C_{ϱ_ψ}, then

$$|\psi(\zeta_1) - \psi(\zeta_2)| \le d(C_{\varrho_\psi}) < d.$$

Hence (by the choice of d) either $\mathcal{A}_{1,\psi}$ or $\mathcal{A}_{2,\psi}$ has diameter $\le \varepsilon$. Since $\varepsilon < \min\{|q_i - q_j|\colon i \ne j\}$, that component cannot contain two of the points q_i. Hence this component must be $\mathcal{A}_{1,\psi} = \psi A_{1,\psi}$. We have proved that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\operatorname{diam} \psi(A) \le \varepsilon$$

whenever $A \subset \partial\mathbb{D}$ has diameter $\operatorname{diam} A \le 2\delta$ and $\psi \in \mathcal{F}_K$. That is, the family $\mathcal{F}_K$ is equicontinuous on $\partial\mathbb{D}$. □

8.21 Solving the Plateau problem

Proof of Theorem 8.1. Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1 Jordan curve and fix a triple (p_1, p_2, p_3) on $\partial\mathbb{D}$ and a triple (q_1, q_2, q_3) on Γ. First we observe that $E_\Gamma < \infty$. Indeed, since $\Gamma \subset \mathbb{R}^3$ is a piecewise C^1 Jordan curve, its has a piecewise C^1 monotone and onto parametrization $\gamma\colon \partial\mathbb{D} \to \Gamma$. Define, by using polar coordinates $r \geq 0$ and $\vartheta \in \partial\mathbb{D}$, a mapping $\omega\colon \bar{\mathbb{D}} \to \mathbb{R}^3$, $\omega(r, \vartheta) = \eta(r)\gamma(\vartheta)$, where $\eta\colon [0,1] \to [0,1]$ is smooth such that $\eta(1) = 1$ and $\eta(r) = 0$ for all $0 \leq r \leq 1/2$. Then ω is Lipschitz, and therefore $E(\omega) < \infty$. Furthermore, $\omega \in X_\Gamma$, and so $E_\Gamma \leq E(\omega) < \infty$. Let then $\psi_j \in X_\Gamma$ be a minimizing sequence for E_Γ, i.e.

$$\lim_{j\to\infty} E(\psi_j) = E_\Gamma.$$

For each j, there exists a harmonic map $u_j\colon \bar{\mathbb{D}} \to \mathbb{R}^3$ such that $u_j \in C(\bar{\mathbb{D}}) \cap C^\infty(\mathbb{D})$ and $u_j|\partial\mathbb{D} = \psi_j|\partial\mathbb{D}$. In particular, $u_j \in X_\Gamma$ and $E(u_j) \leq E(\psi_j)$. Hence also (u_j) is a minimizing sequence. Then we notice that $u_j \circ \varphi$ is harmonic and belongs to X_Γ if $\varphi\colon \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ is a Möbius transformation. Hence by Lemma 8.19, we may assume that each $u_j \in \mathcal{F}_K$ for some fixed $K > 2E_\Gamma$. By the weak compactness Lemma 8.10, there exists a subsequence, still denoted by (u_j), and $\tilde{u} \in W^{1,2}(\mathbb{D})$ such that $u_j \to \tilde{u}$ weakly in $W^{1,2}(\mathbb{D})$ and that

$$E(\tilde{u}) \leq \liminf_{j\to\infty} E(u_j) = E_\Gamma.$$

By Lemma 8.20, (u_j) is equicontinuous on $\partial\mathbb{D}$. Hence, by the Ascoli-Arzelá theorem, there is a subsequence, still denoted by (u_j), that converges uniformly on $\partial\mathbb{D}$. Since each u_j is harmonic, so does $u_i - u_j$, and therefore

$$\sup_{\bar{\mathbb{D}}} |u_i - u_j| = \sup_{\partial\mathbb{D}} |u_i - u_j|$$

by the maximum principle. Hence $u_j \to u$ uniformly on $\bar{\mathbb{D}}$. Now u is a harmonic map as a limit of harmonic mappings in uniform convergence, hence it is smooth in $\mathbb{D}$. Furthermore, $u = \tilde{u}$ (a.e. in $\mathbb{D}$), $u|\partial\mathbb{D}$ is monotone and onto since each $u_j|\partial\mathbb{D}$ does. It follows that $u \in X_\Gamma$, and therefore

$$E(u) = E_\Gamma = A_\Gamma,$$

where the last equation follows from Lemma 8.4. On the other hand, $A_\Gamma \leq \text{Area}(u) \leq E(u)$, and therefore we must have

$$\text{Area}(u) = A_\Gamma.$$

□

References

[CM] T.H. Colding and W.P. Minicozzi II, *A Course in Minimal Surfaces*, Graduate Studies in Mathematics, Vol. 121, AMS, 2011.

[EG] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, 1992.

[GM] M. Giaquinta and L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Edizioni della Normale, Pisa, 2005.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

[G] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, Vol 80, Birkhäuser, 1984.

[MP1] W.H. Meeks and J. Pérez, *The classical theory of minimal surfaces*, Bull. Amer. Math. Soc. 48 (2011), 325-407

[MP2] W.H. Meeks and J. Pérez, *A survey on classical minimal surface theory* www.ugr.es/ jperez/papers/monograph-book2.pdf

[O] R. Osserman, *A survey of Minimal Surfaces*, Dover Publications, 2002.