Department of Mathematics and Statistics
Minimal Surfaces
Exercise 9
28.11.2014

Return by **Thursday, November 27**.

1. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and let $N = (N_1, \ldots, N_{n+1})$ be a unit normal field. Furthermore, let $\delta g = (\delta_1 g, \delta_2 g, \ldots, \delta_{n+1} g)$,
$$\delta g = \nabla g - (\nabla g \cdot N)N,$$
be the tangential gradient of a smooth function g. In the Exercise 8/2 we learned that
$$\delta_i \delta_j - \delta_j \delta_i = \sum_{m=1}^{k+1} \big(N_i \delta_j N_m - N_j \delta_i N_m \big) \tfrac{\partial}{\partial x_m} \quad \forall i, j = 1, \ldots, n+1.$$
Prove that, in fact,
$$\delta_i \delta_j - \delta_j \delta_i = 0 \quad \forall i, j = 1, \ldots, n+1.$$
That is, $[\delta_i, \delta_j] = 0$, as it should be since Σ is a (sub)manifold.

2. Let $\Omega \subset \mathbb{R}^n$ be open. Prove that $\mathrm{BV}(\Omega)$ equipped with the norm
$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \int_\Omega |Du|$$
is a Banach space. [Of course, we identify functions $u, v \in \mathrm{BV}(\Omega)$, with $\|u - v\|_{\mathrm{BV}(\Omega)} = 0$.]

3. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set and let $u \colon \Omega \to \mathbb{R}$ be a continuous function. We define the *relaxed area* of its graph Γ_u as
$$\mathrm{Vol}(\Gamma_u) = \inf \left\{ \liminf_{k \to \infty} \mathcal{V}(u_k) \colon u_k \to u \text{ uniformly in } \Omega,\ u_k \in C^1(\bar{\Omega}) \right\},$$
where the infimum is taken over all such sequences (u_k). Prove that the relaxed area functional $u \mapsto \mathrm{Vol}(\Gamma_u)$ is lower semicontinuous with respect to uniform convergence.
4. Prove that the relaxed area of the graph of a Lipschitz function coincides with its usual area.

5. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set and let $u \in L^1(\Omega)$. We define
$$\mathcal{V}(u) = \inf \left\{ \liminf_{k \to \infty} \mathcal{V}(u_k) \colon u_k \to u \text{ in } L^1(\Omega),\ u_k \in C^1(\bar{\Omega}) \right\}.$$
Prove that
$$\mathcal{V}(u) = \sup \left\{ \int_\Omega \left(u \sum_{i=1}^{n} \tfrac{\partial g_i}{\partial x_i} + g_{n+1} \right) : g \in C_0^1(\Omega; \mathbb{R}^{n+1}),\ |g| \le 1 \right\}.$$