Department of Mathematics and Statistics
Minimal Surfaces
Exercise 8
21.11.2014

Return by **Thursday, November 20**.

Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and let $N = (N_1, \dots, N_{n+1})$ be a unit normal field. Furthermore, let $\delta g = (\delta_1 g, \delta_2 g, \dots, \delta_{n+1} g)$,

$$\delta g = \nabla g - (\nabla g \cdot N) N,$$

be the tangential gradient of a smooth function g.

1. Prove that:
   (a)
   $$\delta_i N_j = \delta_j N_i \quad \forall i, j = 1, \dots, n+1;$$
   (b)
   $$\sum_{i=1}^{n+1} N_i \delta_i = 0;$$
   (c)
   $$\sum_{i=1}^{n+1} N_i \delta_j N_i = 0 \quad \forall j = 1, \dots, n+1.$$

2. Prove that:
   (a)
   $$\delta_i \delta_j - \delta_j \delta_i = \sum_{m=1}^{k+1} \big(N_i \delta_j N_m - N_j \delta_i N_m \big) \tfrac{\partial}{\partial x_m} \quad \forall i, j = 1, \dots, n+1;$$
   (b)
   $$\delta_i \delta_j - \delta_j \delta_i = \sum_{m=1}^{k+1} \big(N_i \delta_j N_m - N_j \delta_i N_m \big) \delta_m \quad \forall i, j = 1, \dots, n+1.$$

3. Prove that
   $$\sum_{i=1}^{n+1} \delta_i \delta_i N_j = -n \delta_j H - N_j \sum_{i,k=1}^{n+1} (\delta_i N_k)^2, \quad \forall j = 1, \dots, n+1,$$
   where H is the mean curvature of Σ with respect to N.

4. Suppose that Σ is the graph of a smooth function $u \colon \Omega \to \mathbb{R}$ and let N be the upwards pointing unit normal field. Furthermore, let $\omega \colon \Omega \times \mathbb{R} \to \mathbb{R}$,
   $$\omega(x, t) = \log \sqrt{1 + |\nabla u(x)|^2} = -\log N_{n+1}.$$
   Prove that
   $$\sum_{i=1}^{n+1} \delta_i \delta_i \omega = \frac{n}{N_{n+1}} \delta_{n+1} H + \sum_{i,j=1}^{n+1} (\delta_i N_j)^2 + \sum_{i=1}^{n+1} (\delta_i \omega)^2.$$