Department of Mathematics and Statistics
Minimal Surfaces
Exercise 7
7.11.2014

Return by **Thursday, November 6**.

1. [McShane-Whitney extension theorem.] Let $X$ be a metric space, $A \subset X$, and $f\colon A \to \mathbb{R}$ an $L$-Lipschitz function. Prove that there exists an $L$-Lipschitz function $\tilde{f}\colon X \to \mathbb{R}$ such that $\tilde{f}|A = f$.

2. Let $(X, d)$ be a metric space and $f\colon X \to [a, b]$ a continuous function. Prove that, for each $i \in \mathbb{N}$, the functions $f_i\colon X \to \mathbb{R}$, $g_i\colon X \to \mathbb{R}$,
$$f_i(x) = \inf\{f(y) + id(x, y)\colon y \in X\},$$
$$g_i(x) = \sup\{f(y) - id(x, y)\colon y \in X\},$$
are $i$-Lipschitz and satisfy
$$a \le f_i(x) \le f_{i+1}(x) \le f(x) \le g_{i+1}(x) \le g_i(x) \le b$$
and
$$\lim_{i\to\infty} f_i(x) = \lim_{i\to\infty} g_i(x) = f(x)$$
for all $x \in X$.

3. Prove that a function $u \in \mathrm{Lip}(\Omega)$ is a minimizer if and only if it is both a superminimizer and a subminimizer (see Definition 6.13).

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $F = (F_1, \ldots, F_n)\colon \Omega \to \mathbb{R}^n$ be a smooth mapping. Furthermore, let $u\colon \bar{\Omega} \to \mathbb{R}$ be a Lipschitz function. Define, for a.e. $x \in \Omega$,
$$\mathrm{div}(uF)(x) = \sum_{i=1}^{n} \frac{\partial(uF_i)}{\partial x_i}(x).$$
   (a) Prove that
   $$\mathrm{div}(uF) = u \,\mathrm{div}\, F + \nabla u \cdot F \quad \text{a.e. in } \Omega.$$
   (b) Suppose that $u = 0$ on $\partial\Omega$ and that $F$ is bounded. Prove that
   $$\int_\Omega \mathrm{div}(uF) = 0.$$