Department of Mathematics and Statistics
Minimal Surfaces
Exercise 5
17.10.2014

Return by **Thursday, October 16**. This time we have only 3 exercises.

1. Prove that Enneper's surface

$$F(s,t) = (s - s^3/3 + st^2, t - t^3/3 + ts^2, s^2 - t^2)$$

intersects itself (is not embedded) but its unit normal (vector field) is asymptotically vertical.

2. A (part of) torus T can be parametrized by

$$F(x_1, x_2) = \big((a + r\cos x_1)\cos x_2, (a + r\cos x_1)\sin x_2, r\sin x_1\big),$$

where $a > r > 0$ and $0 < x_1, x_2 < 2\pi$. Let

$$X_i = \tfrac{\partial F}{\partial x_i} \quad i = 1, 2.$$

Compute:

(a)

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|};$$

(b) the second fundamental forms $\mathrm{II}(X_1, X_1)$, $\mathrm{II}(X_1, X_2)$, and $\mathrm{II}(X_2, X_2)$, and

(c) the Gauss curvature on T.

3. Let f, g be the *Weierstrass-Enneper data* for a minimal surface M. That is, $g\colon D \to \mathbb{C}$ is meromorphic and $f\colon D \to \mathbb{C}$ is analytic in a simply connected domain $D \subset \mathbb{C}$ such that fg^2 is analytic. Let $z = x_1 + ix_2 \mapsto F(z)$ be the parametrization of M given by the Weierstrass-Enneper parametrization and let

$$X_i = \tfrac{\partial F}{\partial x_i} \quad i = 1, 2.$$

Express the basic geometric quantities of M in terms of f and g. More precisely, prove that

(a)

$$g_{ij} = \langle \tfrac{\partial F}{\partial x_i}, \tfrac{\partial F}{\partial x_j} \rangle = \lambda^2 \delta_{ij},$$

where

$$\lambda^2 = \left(\frac{|f|(1 + |g|^2)}{2} \right)^2;$$

(b)

$$X_1 \times X_2 = \frac{|f|^2(1 + |g|^2)}{4} \left(2\,\mathrm{Re}(g), 2\,\mathrm{Im}(g), |g|^2 - 1\right);$$

(c)

$$|X_1 \times X_2| = \lambda^2;$$

(d)

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|} = \frac{1}{|g|^2 + 1} \left(2\,\mathrm{Re}(g), 2\,\mathrm{Im}(g), |g|^2 - 1\right).$$