Department of Mathematics and Statistics
Minimal Surfaces
Exercise 4
10.10.2014

Return by **Thursday, October 9**.

1. Compute the components of the metric

$$g_{ij} = \langle \tfrac{\partial F}{\partial x_i}, \tfrac{\partial F}{\partial x_j} \rangle, \ i, j = 1, 2,$$

   for the following surfaces parametrized by F.
   (a) ellipsoid $F(x_1, x_2) = (a \sin x_1 \cos x_2, b \sin x_1 \sin x_2, c \cos x_1)$;
   (b) elliptic paraboloid $F(x_1, x_2) = (ax_1 \cos x_2, bx_1 \sin x_2, x_2^2)$;
   (c) hyperbolic paraboloid $F(x_1, x_2) = (ax_1 \cosh x_2, bx_1 \sinh x_2, x_2^2)$;
   (d) hyperboloid of two sheets $F(x_1, x_2) = (a \sinh x_1 \cos x_2, b \sinh x_1 \sin x_2, c \cosh x_1)$.

2. Consider a surface of revolution parametrized by

$$F(s, \alpha) = \big(r(s) \cos \alpha, r(s) \sin \alpha, z(s)\big),$$

   where $\big(r'(s)\big)^2 + \big(z'(s)\big)^2 = 1$ for all s. Let

$$X_1 = \tfrac{\partial F}{\partial s} \quad \text{and} \quad X_2 = \tfrac{\partial F}{\partial \alpha}.$$

   Compute the second fundamental forms $\mathrm{II}(X_1, X_1)$, $\mathrm{II}(X_1, X_2)$, and $\mathrm{II}(X_2, X_2)$.

3. Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:
   (a) paraboloid of revolution $\{(x, y, z) \colon z = x^2 + y^2\}$;
   (b) hyperboloid of revolution $\{(x, y, z) \colon x^2 + y^2 - z^2 = 1\}$;
   (c) catenoid $\{(x, y, z) \colon x^2 + y^2 = \cosh^2 z\}$.

4. Consider the parametrized surface (Enneper's surface)

$$F(s, t) = (s - s^3/3 + st^2, t - t^3/3 + ts^2, s^2 - t^2)$$

   and let

$$X_1 = \tfrac{\partial F}{\partial s} \quad \text{and} \quad X_2 = \tfrac{\partial F}{\partial t}.$$

   Show that:
   (a) the components of the metric $g_{ij} = \langle X_i, X_j \rangle$ are

$$g_{11} = g_{22} = (1 + t^2 + s^2)^2, \quad g_{12} = g_{21} = 0;$$

   (b) the scalar second fundamental form (subject to a choice of N) are
   $h(X_1, X_1) = \langle \mathrm{II}(X_1, X_1), N \rangle = 2$, $h(X_1, X_2) = h(X_2, X_1) = 0$, and
   $h(X_2, X_2) = -2$;
   (c) the principal curvatures are

$$\kappa_1 = 2(1 + s^2 + t^2)^{-2} \quad \text{and} \quad \kappa_2 = -2(1 + s^2 + t^2)^{-2}.$$

5. Show that at the origin $(0, 0, 0)$ of the hyperboloid $\{(x, y, z) \colon z = axy\}$, $a > 0$, we have $K = -a^2$ and $H = 0$.