Department of Mathematics and Statistics
Minimal Surfaces
Exercise 3
3.10.2014

Return by **Thursday, October 2**.

In the pdf-file Pg6-7.pdf there is a copy of the proof of the first variation formula (taken from [CM]). There are three points in the proof that require more attention. Discuss about these and try to find explanations.

1. The definition of $\nu(t)$. Since x_i are local coordinates on Σ, what does $g_{ij}(t)$ mean for $t \neq 0$. See also the integral in (1.40) that is taken over Σ.

2. Why $\nu(t)$ is independent of local coordinates on Σ?

3. Why the t and x_i derivatives commute, i.e. $[F_t, F_{x_i}] = 0$?

4. Let $M^n \subset \mathbb{R}^m$ be a smooth submanifold. Suppose that (U, x), $x = (x^1, x^2, \dots, x^n)$, is a chart on M and let $F = x^{-1} \colon xU \to U$ be a parametrization of U. Verify that
$$\int_U \sqrt{g} dx^1 \wedge \cdots \wedge dx^n = \int_{xU} \sqrt{\det f F^* dF} dm,$$
where $g = \det(g_{ij})$, $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$, $dF(p) \colon \mathbb{R}^n \to \mathbb{R}^m$ is the differential of F at p and $dF^* \colon \mathbb{R}^m \to \mathbb{R}^n$ is the adjoint of $dF(p)$.

5. Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Show that $\mathbb{S}^n$ has constant sectional curvature 1, i.e. $K(P) = 1$ for all 2-dimensional subspaces $P \subset T_p\mathbb{S}^n$, $\forall p \in \mathbb{S}^n$.

6. Let $f \colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function,
$$M = \{x \in \mathbb{R}^{n+1} \colon f(x) = a\} \neq \emptyset,$$
and suppose that $\nabla f(x) \neq 0$ $\forall x \in M$. Prove that (up to the sign) the mean curvature of M is
$$H = -\frac{1}{n} \operatorname{div} \left(\frac{\nabla f}{|\nabla f|} \right).$$