Department of Mathematics and Statistics
Minimal Surfaces
Exercise 2
26.9.2014

Return by **Thursday, September 25**.

1. Let $\bar{\nabla}$ be the standard connection of $\mathbb{R}^m$ and $X, Y \in \mathcal{T}(\mathbb{R}^m)$ smooth vector fields. Verify that $\big(\bar{\nabla}_X Y\big)_p$ depends only on the vector X_p and values of Y along a path $\gamma\colon]-\varepsilon,\varepsilon[\to \mathbb{R}^m$, with $\gamma(0) = p$ and $\dot\gamma_0 = \gamma'(0) = X_p$.

2. Let $M \subset \mathbb{R}^m$ be a smooth n-dimensional submanifold of $\mathbb{R}^m$ and let $p \in M$. Suppose further that $v_1, v_2 \ldots, v_n$ is a basis of T_pM. Prove that the mean curvature vector at p is given by
$$H_p = \sum_{i,j=1}^{n} g^{ij}\,\mathrm{II}_p(v_i, v_j),$$
where (g^{ij}) is the inverse of the matrix (g_{ij}), $g_{ij} = \langle v_i, v_j\rangle$.

3. Verify that the *Riemannian curvature tensor*
$$\bar{R}\colon \mathcal{T}(\mathbb{R}^m)\times\mathcal{T}(\mathbb{R}^m)\times\mathcal{T}(\mathbb{R}^m)\to\mathcal{T}(\mathbb{R}^m),$$
$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z$$
vanishes identically.

4. Prove the Gauss equation for smooth submanifolds of $\mathbb{R}^m$:
$$K(P)|v\wedge w|^2 = \langle \mathrm{II}_p(v,v), \mathrm{II}_p(w,w)\rangle - |\,\mathrm{II}_p(v,w)|^2.$$

5. Suppose that $M \subset \mathbb{R}^m$ is a smooth submanifold, $\gamma\colon I \to M$ a smooth path, and V a smooth vector fiels along γ tangent to M (i.e. $V_t \in T_{\gamma(t)}M\ \forall t$). Show that $(D_tV)_t$ is the orthogonal projection onto $T_{\gamma(t)}M$ of the ordinary Euclidean derivative $\dot{V}_t = V'(t)$. Furthermore, show that γ is a geodesic on M (i.e. $D_tV \equiv 0$) if and only if its Euclidean accelleration $\ddot\gamma$ is everywhere normal to M.

6. Suppose that Ω is an open subset of $\mathbb{R}^2$ and $f\colon \Omega\to\mathbb{R}$ a smooth function. Let M be the graph of f and N the upward-pointing unit normal vector field on M. Then M is parametrized by *graph coordinates* $(x,y)\in\Omega$,
$$(x,y)\mapsto \big(x,y,f(x,y)\big).$$
Compute the components of the Weingarten map in graph coordinates, in terms of f and its partial derivatives.