Department of Mathematics and Statistics
Minimal Surfaces
Exercise 10
12.12.2014

Return by **Thursday, December 11**.

In Exercises 1-5 we will prove (step by step) the mean value property for weakly harmonic functions.
Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be an open set. A function $u \in L^1_{\text{loc}}(\Omega)$ is *weakly harmonic* if

$$\int_\Omega u(x)\Delta v(x) = 0 \quad \forall v \in C_0^\infty(\Omega). \tag{1}$$

1. Let $x \in \Omega$ and $R > 0$ such that $B^n(x,R) \subset \Omega$. Choose $\varepsilon \in (0,R)$ and $\varphi \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset (\varepsilon, R)$. Define

$$r(y) = |x-y|$$

and

$$v(y) = \varphi\big(r(y)\big). \tag{2}$$

Prove that:
(a) $v \in C_0^\infty(\Omega)$,
(b)

$$\Delta v(y) = \varphi''\big(r(y)\big) + \frac{n-1}{r(y)}\varphi'\big(r(y)\big),$$

(c)

$$\varphi''(r) + \frac{n-1}{r}\varphi'(r) = r^{1-n}\frac{d}{dr}\left(r^{n-1}\varphi'(r)\right).$$

2. Let φ be as above and define

$$\psi(r) = \frac{d}{dr}\left(r^{n-1}\varphi'(r)\right).$$

(a) Prove that $\psi \in C_0^\infty\big((\varepsilon,R)\big)$ and

$$\int_\varepsilon^R \psi(r)\,dr = 0.$$

(b) Prove the converse: That is, given $\psi \in C_0^\infty\big((\varepsilon,R)\big)$ satisfying

$$\int_\varepsilon^R \psi(r)\,dr = 0, \tag{3}$$

find $\varphi \in C_0^\infty\big((\varepsilon,R)\big)$ such that

$$\psi(r) = \frac{d}{dr}\left(r^{n-1}\varphi'(r)\right). \tag{4}$$

3. Let $\psi \in C_0^\infty\big((\varepsilon, R)\big)$ be an arbitrary function satisfying (3). Then define φ and v as above (in (4) and (2)). Apply (1) with this v and verify that

$$\int_\varepsilon^R \psi(r)\omega(r)\,dr = 0, \tag{5}$$

where

$$\omega(r) = r^{1-n}\int_{\partial B^n(x,r)} u\,d\sigma$$

and σ is the (normalized) $(n-1)$-dimensional Hausdorff measure.

4. Prove that $\omega(r)$ is constant for a.e. $r \in (\varepsilon, R)$. Note that this constant depends on the chosen $x \in \Omega$. Denote this constant by $\sigma_{n-1}\bar{u}(x)$, where $\sigma_{n-1} = \sigma(\partial B^n(0,1))$. Verify then that

$$\bar{u}(x) = \frac{1}{\sigma_{n-1}r^{n-1}}\int_{\partial B(x,r)} u\,d\sigma$$

for a.e. $r \in (0, R)$.

5. Prove that

$$\bar{u}(x) = \frac{1}{m\big(B^n(x,R)\big)}\int_{B^n(x,R)} u\,dx$$

for every $x \in \Omega$ and $R > 0$ such that $B^n(x,R) \subset \Omega$. Furthermore, prove that $\bar{u}$ is continuous in Ω and $u = \bar{u}$ a.e. in Ω.

6. Let $\mathbb{D} \subset \mathbb{R}^2$ be the unit disc and $u \in C(\bar{\mathbb{D}};\mathbb{R}^3) \cap W^{1,2}(\mathbb{D};\mathbb{R}^3)$. Let $u_k \in C^\infty(\bar{\mathbb{D}};\mathbb{R}^3) \cap W^{1,2}(\mathbb{D};\mathbb{R}^3)$ be a sequence of mappings such that $u_k \to u$ uniformly in $\bar{\mathbb{D}}$ and $u_k \to u$ in $W^{1,2}(\mathbb{D};\mathbb{R}^3)$.[1] For each $k \in \mathbb{N}$, let

$$\begin{aligned}
u_k &= (u_k^1, u_k^2, u_k^3),\\
\frac{\partial u_k}{\partial x} &= \left(\frac{\partial u_k^1}{\partial x}, \frac{\partial u_k^2}{\partial x}, \frac{\partial u_k^3}{\partial x}\right),\\
\frac{\partial u_k}{\partial y} &= \left(\frac{\partial u_k^1}{\partial y}, \frac{\partial u_k^2}{\partial y}, \frac{\partial u_k^3}{\partial y}\right),\\
g_{xx}^{(k)} &= \frac{\partial u_k}{\partial x}\cdot\frac{\partial u_k}{\partial x},\\
g_{xy}^{(k)} &= g_{xy}^{(k)} = \frac{\partial u_k}{\partial x}\cdot\frac{\partial u_k}{\partial y},\\
g_{yy}^{(k)} &= \frac{\partial u_k}{\partial y}\cdot\frac{\partial u_k}{\partial y},\\
g^{(k)} &= \det\begin{pmatrix} g_{xx}^{(k)} & g_{xy}^{(k)} \\ g_{yx}^{(k)} & g_{yy}^{(k)}\end{pmatrix}.
\end{aligned}$$

Define $g_{xx}, g_{xy}, g_{yx}, g_{yy}$, and g similarly for the mapping u.

Study what happens to the integrals

$$\int_{\mathbb{D}} \sqrt{g^{(k)}}\,dxdy$$

as $k \to \infty$.

[1] See [EG], Theorems 1-3 in Section 4.2.