Minimal submanifolds

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Overview

Area functional

Suppose that $u\colon \bar{\Omega} \to \mathbb{R}$ is a C^2-function, where $\Omega \subset \mathbb{R}^2$ is a bounded open set. Denote by $\Sigma = \Sigma_u \subset \mathbb{R}^3$ its graph

$$\Sigma = \big\{ \big(x, y, u(x, y)\big) \colon (x, y) \in \bar{\Omega} \big\}.$$

It is a 2-dimensional submanifold of $\mathbb{R}^3$ and the tangent space (plane) $T_p\Sigma$ at $p = (x, y, u(x, y)) \in \Sigma$ is spanned by vectors $(1, 0, u_x)$ and $(0, 1, u_y)$, where u_x and u_y denote the partial derivatives of u with respect to x and y, respectively.
The absolute value

$$\big|(1, 0, u_x) \times (0, 1, u_y)\big|$$

is the area of the parallelogram spanned by $(1, 0, u_x)$ and $(0, 1, u_y)$, and so the area of the graph is

Area functional

$$\mathcal{A}(\Sigma) = \int_\Omega |(1,0,u_x) \times (0,1,u_y)| = \int_\Omega \sqrt{1+u_x^2+u_y^2}$$
$$= \int_\Omega \sqrt{1+|\nabla u|^2}.$$

Let $\eta \in C_0^2(\Omega)$. Then the graphs of u and $u+t\eta,\ t\in\mathbb{R}$, have the same "boundary" $\partial\Sigma = \{(x,y,u(x,y)\colon (x,y)\in\partial\Omega\}$ and

$$\mathcal{A}(\Sigma_{u+t\eta}) = \int_\Omega \sqrt{1+|\nabla u + t\nabla\eta|^2}.$$

Suppose that Σ_u has the minimal area among all graphs with the same boundary $\partial\Sigma_u$. Then, of course,

$$\frac{d}{dt}\mathcal{A}(\Sigma_{u+t\eta})_{|t=0} = 0.$$

Minimal graph equation

Differentiating with respect to t and using Green's formula we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{A}(\Gamma_{u+t\eta})_{|t=0} &= \frac{d}{dt}\int_\Omega \sqrt{1+|\nabla u + t\nabla\eta|^2}_{|t=0} \\
&= \int_\Omega \frac{d}{dt}\sqrt{1+|\nabla u + t\nabla\eta|^2}_{|t=0} \\
&= \int_\Omega \frac{1}{2}(1+|\nabla u|^2)^{-1/2}\frac{d}{dt}\big\langle\nabla(u+t\eta),\nabla(u+t\eta)\big\rangle_{|t=0} \\
&= \int_\Omega \frac{\langle\nabla u,\nabla\eta\rangle}{\sqrt{1+|\nabla u|^2}} \\
&= -\int_\Omega \eta\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \\
&= 0.
\end{aligned}$$

Minimal graph equation

We say that $u \in C^2(\Omega)$ is a *critical point for the area functional* if

$$\frac{d}{dt}\mathcal{A}(\Sigma_{u+t\eta})_{|t=0} = 0 \quad \forall \eta \in C_0^2(\Omega).$$

In that case, since

$$\int_\Omega \eta \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

for all $\eta \in C_0^2(\Omega)$, we conclude that $u \in C^2(\Omega)$ is a critical point if and only if it satisfies the *minimal graph equation* (or the *mean curvature equation*)

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

Minimal graph equation

In particular, if $u \in C^2(\Omega)$ minimizes the area (of graphs) among smooth functions with the same values on $\partial\Omega$, it is a solution to the minimal graph equation.
Conversely, a critical point u for the area functional minimizes the area among all (smooth) surfaces *inside the cylinder* $\Omega \times \mathbb{R}$ with the same boundary $\partial\Sigma_u$.
For this and later purposes we note that the unit vector

$$N = \frac{(1,0,u_x) \times (0,1,u_y)}{|(1,0,u_x) \times (0,1,u_y)|} = \frac{(-u_x,-u_y,1)}{\sqrt{1+|\nabla u|^2}}$$

$$= \frac{(\overbrace{-\nabla u}^{\in\mathbb{R}^2},1)}{\sqrt{1+|\nabla u|^2}}$$

is orthogonal to both $(1,0,u_x)$ and $(0,1,u_y)$, and therefore it is the (upwards pointing) unit normal to Σ_u.

Minimal graph equation

We define a 2-form ω in the cylinder $\Omega \times \mathbb{R}$ by setting

$$\omega(X, Y) = \det(X, Y, N)$$

for vectors $X, Y \in \mathbb{R}^3$.
Note that ω is the contraction by N of the standard volume form $\tilde{\omega} = dx \wedge dy \wedge dz$, i.e. $\omega = N \lrcorner \tilde{\omega} = i_N \tilde{\omega}$. Hence ω is the volume (area) form of Σ_u.
Since $\omega = a\, dx \wedge dy + b\, dx \wedge dz + c\, dy \wedge dz$ and

$$a = \omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 1/\sqrt{1 + |\nabla u|^2},$$
$$b = \omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = u_y/\sqrt{1 + |\nabla u|^2},$$
$$c = \omega(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = -u_x/\sqrt{1 + |\nabla u|^2},$$

Minimal graph equation

we see that

$$\omega = \frac{dx \wedge dy - u_x dy \wedge dz - u_y dz \wedge dx}{\sqrt{1 + |\nabla u|^2}}.$$

Furthermore, since u satisfies the minimal graph equation, we obtain

$$d\omega = \left\{ \frac{\partial}{\partial x} \left(\frac{-u_x}{\sqrt{1 + |\nabla u|^2}} \right) + \frac{\partial}{\partial y} \left(\frac{-u_y}{\sqrt{1 + |\nabla u|^2}} \right) \right\} dx \wedge dy \wedge dz$$
$$= 0.$$

Thus ω is a closed 2-form in the cylinder $\Omega \times \mathbb{R}$.
Let then Σ be another (smooth) surface (not necessarily a graph) in $\Omega \times \mathbb{R}$ with the same boundary than Σ_u ($\partial\Sigma_u = \partial\Sigma$). Then Σ and Σ_u bound an open set $U \subset \mathbb{R}^3$ where $d\omega = 0$. The set U may have several components but applying Stokes' theorem in each component we obtain

$$\int_{\Sigma_u} \omega = \int_{\Sigma} \omega.$$

On the other hand, by definition $|\omega(X, Y)| = |\det(X, Y, N)|$ is the volume of the polyhedron spanned by vectors X, Y, and N. In particular, for any unit vectors X and Y,

$$|\omega(X, Y)| \leq 1,$$

with the equality if and only if X, Y, and N are orthonormal. Hence

$$\mathcal{A}(\Sigma_u) = \int_{\Sigma_u} \omega = \int_{\Sigma} \omega \leq \mathcal{A}(\Sigma). \tag{1}$$

This shows that Σ_u minimizes the area among such surfaces (inside $\Omega \times \mathbb{R}$).
If Ω is convex, then Σ_u is area-minimizing among all surfaces $\Sigma \subset \mathbb{R}^3$ with $\partial\Sigma = \partial\Sigma_u$. To see this, let Σ be such a surface and let $P\colon \mathbb{R}^3 \to \Omega \times \mathbb{R}$ be the nearest point projection. The convexity of Ω implies that P is 1-Lipschitz map that is equal to the identity on $\Omega \times \mathbb{R}$. In particular, $\mathcal{A}(P\Sigma) \leq \mathcal{A}(\Sigma)$. Applying (1) to $P\Sigma$ we obtain

$$\mathcal{A}(\Sigma_u) \leq \mathcal{A}(P\Sigma) \leq \mathcal{A}(\Sigma).$$

Minimal graph equation

Remark

All of the above holds in higher dimensions, too.

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u\colon \bar{\Omega} \to \mathbb{R}$ a smooth function. Then the area (n-dimensional measure) of the graph

$$\Sigma_u = \big\{ \big(x, u(x)\big) \colon x \in \Omega \big\} \subset \Omega \times \mathbb{R}$$

is:

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx.$$

Remark

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a solution to the minimal graph equation, its graph Σ_u need *not* minimize the area among *all* hypersurfaces with the same boundary $\partial \Sigma_u$. (Hardt, Lau, Lin: *Non-minimality of minimal graphs*, Indiana Univ. Math. J. 36 (1987), 849-855)

Standard connection of $\mathbb{R}^m$

We denote by

$$\partial_i = \frac{\partial}{\partial x_i}, \; i = 1, \dots, m,$$

the standard basis of $\mathbb{R}^m$. Thus these vectors are orthonormal with respect to the standard inner product $\langle \cdot, \cdot \rangle$.
A vector field defined on an open set $\Omega \subset \mathbb{R}^m$ is a mapping $V\colon \Omega \to \mathbb{R}^m$ which we write as

$$V_p = V(p) = \sum_{i=1}^{m} v^i(p) \partial_i,$$

where $v^i\colon \Omega \to \mathbb{R}, \; i = 1, \dots, m$, are (component) functions.
Vector fields act on smooth functions f as

$$Vf = \sum_{i=1}^{m} v^i(p) \partial_i f, \quad \partial_i f = \frac{\partial f}{\partial x_i}.$$

Standard connection of $\mathbb{R}^m$

Thus

$$V_p f := Vf(p) = \sum_{i=1}^{m} v^i(p)\partial_i f(p) = \langle V_p, \nabla f(p)\rangle$$

is the *directional derivative* of f along vector V_p.

Definition

Let X and V be vector fields such that V is smooth (i.e. the component functions v^i are smooth). Then the *covariant derivative* of V in the direction X_p is the vector

$$\big(\bar{\nabla}_X V\big)_p = (X_p v^1, X_p v^2, \ldots, X_p v^m) \in \mathbb{R}^m$$

and $\bar{\nabla}_X V$ is the vector field $p \mapsto \big(\bar{\nabla}_X V\big)_p$.

We denote by $\mathcal{T}(\Omega)$ the set of all smooth vector fields on $\Omega \subset \mathbb{R}^m$.

Standard connection of $\mathbb{R}^m$

Definition

The mapping

$$\bar{\nabla}\colon \mathcal{T}(\Omega) \times \mathcal{T}(\Omega) \to \mathcal{T}(\Omega), \quad \bar{\nabla}(X, Y) = \bar{\nabla}_X Y,$$

is called the *Levi-Civita connection* on Ω. We also call it the standard connection on $\Omega \subset \mathbb{R}^m$.

The standard connection has the following properties:

1. $\bar{\nabla}_X Y$ is C^∞-linear in X: for every functions $f, g \in C^\infty(\Omega)$ and vector fields $X, Y, V \in \mathcal{T}(\Omega)$

$$\bar{\nabla}_{fX+gY} V = f\bar{\nabla}_X V + g\bar{\nabla}_Y V;$$

2. $\bar{\nabla}_X Y$ is $\mathbb{R}$-linear in Y: for every $a, b \in \mathbb{R}, X, Y, V \in \mathcal{T}(\Omega)$

$$\bar{\nabla}_X(aY + bV) = a\bar{\nabla}_X Y + b\bar{\nabla}_X V;$$

Standard connection of $\mathbb{R}^m$

3. $\bar{\nabla}$ satisfies the Leibniz rule: for every $f \in C^\infty(\Omega), X, Y \in \mathcal{T}(\Omega)$

$$\bar{\nabla}_X(fY) = f\bar{\nabla}_X Y + (Xf)Y;$$

4. $\bar{\nabla}$ is *torsion-free*: for every $X, Y \in \mathcal{T}(\Omega)$

$$\bar{\nabla}_X Y - \bar{\nabla}_Y X = [X, Y],$$

   where $[X, Y] \in \mathcal{T}(\Omega)$ is the Lie bracket

$$[X, Y]f = X(Yf) - Y(Xf);$$

5. $\bar{\nabla}$ is *compatible* with the standard inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^m$: for every $X, Y, Z \in \mathcal{T}(\Omega)$

$$X\langle Y, Z\rangle = \langle \bar{\nabla}_X Y, Z\rangle + \langle Y, \bar{\nabla}_X Z\rangle.$$

The standard connection $\bar{\nabla}$ is the unique mapping $\mathcal{T}(\Omega) \times \mathcal{T}(\Omega) \to \mathcal{T}(\Omega)$ satisfying the properties above.

Riemannian metric on a submanifold of $\mathbb{R}^{n+k}$

Let $\Omega \subset \mathbb{R}^n$ be an open set and $\varphi\colon \Omega \to \mathbb{R}^m$ a smooth mapping. We say that φ is an *immersion* if the differential $d\varphi(x)\colon \mathbb{R}^n \to \mathbb{R}^m$ is injective for all $x \in \Omega$. Then necessarily $m \geq n$.

If φ is one-to-one, the image $M = \varphi\Omega \subset \mathbb{R}^m$ is called an *immersed submanifold* of $\mathbb{R}^m$.

If, in addition, φ is a homeomorphism onto $\varphi\Omega \subset \mathbb{R}^m$, then φ is an *embedding* and $M = \varphi\Omega$ is an n-dimensional submanifold of $\mathbb{R}^m$. Note that here M has the relative topology.

In general, a smooth manifold $M \subset \mathbb{R}^m$ is a *submanifold* of $\mathbb{R}^m$ if the inclusion $\pi\colon M \hookrightarrow \mathbb{R}^m,\ \pi(x) = x$, is an embedding. [We use the notation $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ for the inclusion, because then $\pi_i\colon M \to \mathbb{R}$ will be the projection to the x_i-axis.]

Let $M \subset \mathbb{R}^m$ be a smooth n-dimensional submanifold of $\mathbb{R}^m$. Thus locally M can be parametrized by a smooth homeomorphism $\varphi\colon \Omega \to U$, where $\Omega \subset \mathbb{R}^n$ and $U \subset M$ are open, and the differential $d\varphi(x)$ at x is of rank n for every $x \in \Omega$.

Riemannian metric on a submanifold of $\mathbb{R}^{n+k}$

We identify the tangent space $T_pM, p \in U$, with the image $d\varphi(\varphi^{-1}(p))\mathbb{R}^n$. Thus T_pM is an n-dimensional vector subspace of $\mathbb{R}^m$. Each T_pM inherits an inner product $\langle \cdot, \cdot \rangle$ from $\mathbb{R}^m$: for every vectors $v, w \in T_pM$,

$$\langle v, w \rangle = v \cdot w,$$

where $v \cdot w$ is just the standard inner product in $\mathbb{R}^m$. This induced inner product $\langle \cdot, \cdot \rangle$ defines the Riemannian metric (and thus the Riemannian submanifold structure) on M.
For every $p \in M$, the inner product of $\mathbb{R}^m$ splits $\mathbb{R}^m$ orthogonally into

$$T_pM \oplus T_pM^{\perp}.$$

We write $N_pM = T_pM^{\perp}$ and call it the *normal space* of M at p. Furthermore, we denote by

$$TM = \bigsqcup_{p \in M} T_pM \quad \text{and} \quad NM = \bigsqcup_{p \in M} N_pM$$

the tangent and normal bundles, respectively.

Levi-Civita connection on a submanifold of $\mathbb{R}^{n+k}$

Next we define the Levi-Civita connection ∇ on M that satisfies conditions 1.-5. above, in particular, that is compatible with the induced Riemannian metric.
Let $\tilde{X}, \tilde{Y} \in \mathcal{T}(\Omega)$ be smooth vector fields in an open set $\Omega \subset \mathbb{R}^m$. Then at every $p \in \Omega \cap M$

$$(\bar{\nabla}_{\tilde{X}} \tilde{Y})_p = (\bar{\nabla}_{\tilde{X}} \tilde{Y})_p^{\top} + (\bar{\nabla}_{\tilde{X}} \tilde{Y})_p^{\perp},$$

where

$$(\bar{\nabla}_{\tilde{X}} \tilde{Y})_p^{\top} \in T_pM \quad \text{and} \quad (\bar{\nabla}_{\tilde{X}} \tilde{Y})_p^{\perp} \in N_pM.$$

Levi-Civita connection on a submanifold of $\mathbb{R}^{n+k}$

Definition

The Levi-Civita connection ∇ of M is simply the orthogonal projection on TM of the standard connection of $\mathbb{R}^m$. More precisely, let $X, Y \in \mathcal{T}(U)$ be smooth vector fields on an open set $U \subset M$, i.e. at each point $p \in U$

$$X_p = \sum_{i=1}^{m} a^i(p)\partial_i, \quad Y_p = \sum_{i=1}^{m} b^i(p)\partial_i,$$

where $a^i, b^i \colon U \to \mathbb{R}$ are smooth functions. For each $p \in U$, let $\tilde{X}$ and $\tilde{Y}$ be (any) smooth extensions of X and Y to a neighborhood (in $\mathbb{R}^m$) of p. Then we define

$$(\nabla_X Y)_p = \left(\bar{\nabla}_{\tilde{X}} \tilde{Y}\right)_p^{\top} \in T_pM,$$

Levi-Civita connection on a submanifold of $\mathbb{R}^{n+k}$

where

$$(\bar{\nabla}_{\tilde{X}}\tilde{Y})_p^{\top}$$

is the orthogonal projection of $(\bar{\nabla}_{\tilde{X}}\tilde{Y})_p$ to T_pM.

Remark

The properties 1.-5. hold for ∇. In particular, ∇ is torsion-free and compatible with the induced inner product (Riemannian metric).

Remark

Note that $\nabla_X Y$ is well-defined, i.e. does not depend on the extensions $\tilde{X}$ and $\tilde{Y}$.

Second fundamental form of M

Denote by $\mathcal{N}(M)$ the set of all smooth mappings $V\colon M \to \mathbb{R}^m$ such that $V_p \in N_pM$ for all $p \in M$.

Definition

The *second fundamental form* of M is the map
$\mathrm{B}\colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{N}(M)$,

$$\mathrm{B}(X, Y) = \left(\bar{\nabla}_{\tilde{X}} \tilde{Y}\right)^{\perp},$$

where $\tilde{X}$ and $\tilde{Y}$ are smooth extensions of X and Y, respectively.

Thus we have the *Gauss formula* on M

$$\bar{\nabla}_X Y = \nabla_X Y + \mathrm{B}(X, Y)$$

for vector fields $X, Y \in \mathcal{T}(M)$.

Second fundamental form of M

Note again that the left hand side makes sense since $\left(\bar{\nabla}_X Y\right)_p$ depends only on $X_p \in T_pM$ and values of Y along any path $\gamma\colon]-\varepsilon,\varepsilon[\to M$, with $\gamma(0)=p$ and $\dot\gamma_0 = X_p$.

Lemma

The second fundamental form is

(a) independent of extensions of X and Y;

(b) symmetric in X and Y;

(c) C^∞-bilinear.

Lemma [The Weingarten equation]

Suppose $X, Y \in \mathcal{T}(M)$ and $N \in \mathcal{N}(M)$. Then on M we have

$$\langle \bar{\nabla}_X N, Y\rangle = -\langle N, \mathrm{B}(X,Y)\rangle,$$

where X, Y, and N are extended to $\mathbb{R}^m$ (and still denoted by X, Y, N).

Mean curvature vector

Definition

The *mean curvature vector* H of M at $p \in M$ is ("the trace of the second fundamental form")

$$H_p = \sum_{i=1}^{n} \mathrm{B}(X_i, X_i),$$

where $X_1, \ldots, X_n$ is an orthonormal basis of T_pM.

In general, if $v_1, v_2, \ldots, v_n$ is an arbitrary basis of T_pM and $g_{ij} = \langle v_i, v_j \rangle$, then

$$H_p = \sum_{i,j=1}^{n} g^{ij}\, \mathrm{B}(v_i, v_j),$$

where (g^{ij}) is the inverse of the matrix (g_{ij}).

Mean curvature vector

Remarks

Note that $H_p \in N_pM$.
Often H_p is defined as ("the mean trace of the second fundamental form")

$$H_p = \frac{1}{n}\sum_{i=1}^{n} \mathrm{B}(X_i, X_i),$$

where $X_1, \ldots, X_n$ is an orthonormal basis of T_pM.

Definition

An immersed submanifold $M \subset \mathbb{R}^m$ is *minimal* if $H \equiv 0$ on M.

Scalar second fundamental form

Let M be an $(m-1)$-dimensional submanifold of $\mathbb{R}^m$, i.e. a *hypersurface*.

Definition

The *scalar second fundamental form* of M is the symmetric 2-tensor defined by

$$h(X, Y) = \langle \mathrm{B}(X, Y), N \rangle,$$

where $N \in \mathcal{N}(M)$ is a smooth unit normal vector field.

Since M is of co-dimension 1, the unit normal vector N_p spans N_pM at every point $p \in M$. Hence

$$\mathrm{B}(X, Y) = h(X, Y)N.$$

Note that the sign of h depends on the choice of N (versus $-N$).
We have the *Gauss formula* for hypersurfaces of $\mathbb{R}^m$:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)N.$$

Weingarten map

Definition

The *Weingarten map* $L\colon TM \to TM$ is defined as

$$LX = -\bar{\nabla}_X N.$$

Lemma

For each $p \in M$, the Weingarten map is a self-adjoint endomorphism of T_pM.

Since for every $p \in M$, $L\colon T_pM \to T_pM$ is self-adjoint, it follows from linear algebra that it has real eigenvalues $\kappa_1, \kappa_2, \ldots, \kappa_{m-1}$ and that there exists an orthonormal basis $E_1, E_2, \ldots, E_{m-1}$ of T_pM consisting of eigenvectors

$$LE_i = \kappa_i E_i, \quad i = 1, \ldots, m-1.$$

The eigenvalues of L are called the *principal curvatures* and the corresponding eigenvectors are called *principal directions*.

Weingarten map

Let $\kappa_1, \kappa_2, \ldots, \kappa_{m-1}$ and $E_1, E_2, \ldots, E_{m-1}$ be as above. By the Weingarten equation

$$\langle N, \mathrm{B}(E_i, E_i)\rangle = -\langle \bar{\nabla}_{E_i} N, E_i\rangle = \langle LE_i, E_i\rangle = \langle \kappa_i E_i, E_i\rangle = \kappa_i.$$

Hence the mean curvature vector is given by

$$H_p = \Big(\sum_{i=1}^{m-1} \kappa_i\Big) N.$$

The *Gaussian curvature* of M at p is the determinant

$$K = \det L = \kappa_1 \kappa_2 \cdots \kappa_{m-1}.$$

Riemannian curvature tensor

Definition

The *Riemannian curvature tensor* of M is the mapping

$$R\colon \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$$

defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Note that the Riemannian curvature tensor $\bar{R}$ of $\mathbb{R}^m$ vanishes identically.

The *sectional curvature* of a 2-dimensional subspace $P \subset T_p M$ spanned by vectors $v, w \in T_p M$ is defined by

$$K^M(P) = \frac{\langle R^M(v, w)w, v \rangle}{|v \wedge w|^2},$$

where

Riemannian curvature tensor

$$|v \wedge w| = \sqrt{|v|^2|w|^2 - \langle v, w\rangle^2}$$

is the area of the parallelogram spanned by v and w.
It satisfies the *Gauss equation*

$$K^M(P)|v \wedge w|^2 - \underbrace{\bar{K}(P)|v \wedge w|^2}_{=0} = \langle \mathrm{B}(v, v), \mathrm{B}(w, w)\rangle - |\,\mathrm{B}(v, w)|^2.$$

Here $\bar{K}(P)$ denotes the sectional curvature of P with respect to the ambient space which in our setting is $\mathbb{R}^m$ and therefore $\bar{K} \equiv 0$.

Riemannian curvature tensor

Let $\kappa_1, \kappa_2, \ldots, \kappa_{m-1}$ and $E_1, E_2, \ldots, E_{m-1}$ be as above. By the Weingarten equation

$$\langle N, \mathrm{B}(E_i, E_j)\rangle = -\langle \bar{\nabla}_{E_i} N, E_j\rangle = \langle LE_i, E_j\rangle = \langle \kappa_i E_i, E_j\rangle = \kappa_i \delta_{ij}.$$

Hence

$$\mathrm{B}(E_i, E_j) = \kappa_i \delta_{ij} N$$

and therefore

$$K(P) = \langle \mathrm{B}(E_i, E_i), \mathrm{B}(E_j, E_j)\rangle - \underbrace{|\,\mathrm{B}(E_i, E_j)|^2}_{=0} = \kappa_i \kappa_j$$

for a 2-dimensional subspace $P = \mathrm{span}(E_i, E_j) \subset T_pM$.

Gradient

Let $M \subset \mathbb{R}^m$ be an n-dimensional smooth submanifold. Let $f\colon M \to \mathbb{R}$ be a C^1-function, $p \in M$, and $X \in T_pM$. Then

$$Xf = (f \circ \gamma)'(0),$$

where $\gamma\colon]-\varepsilon,\varepsilon[\to M$ is any C^1-path, with $\gamma(0) = p$ and $\dot\gamma_0 = X$.
The *gradient* of f is defined as

$$\nabla^M f(p) = \sum_{i=1}^{n} (X_i f) X_i,$$

where $\{X_i\}_{i=1}^n$ is an orthonormal basis of T_pM.
In particular, if f is a C^1-function in a neighborhood (in $\mathbb{R}^m$) of p, then

$$\nabla^M f(p) = \big(\nabla f(p)\big)^\top,$$

where

$$\nabla f(p) = \sum_{i=1}^{m} \partial_i f(p) \partial_i$$

Gradient

is the standard gradient (in $\mathbb{R}^m$) of f.
Given a chart $\varphi\colon U \to \mathbb{R}^n,\ U \subset M$, and the corresponding local parametrization $F = \varphi^{-1}\colon \varphi U \to U$ we can write $\nabla^M f$ in U as

$$\nabla^M f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial F}{\partial x^j},$$

where $g^{ij}\colon U \to \mathbb{R},\ \frac{\partial f}{\partial x^i}\colon U \to \mathbb{R}$, and $\frac{\partial F}{\partial x^j}\colon U \to TM$ are defined as

$$\begin{aligned}
\frac{\partial f}{\partial x^i}(p) &= \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)),\\
\frac{\partial F}{\partial x^j}(p) &= \left(\frac{\partial F_1}{\partial x^j}(\varphi(p)), \ldots, \frac{\partial F_m}{\partial x^j}(\varphi(p))\right) \in T_pM,\\
g_{ij}(p) &= \frac{\partial F}{\partial x^i}(p) \cdot \frac{\partial F}{\partial x^j}(p),
\end{aligned}$$

and (g^{ij}) is the inverse of the matrix (g_{ij}).

Divergence

The *divergence* (on M) of a C^1-smooth vector field V (not necessarily tangential) at $p \in M$ is defined as follows.
Let $\{X_1, X_2, \ldots, X_n, Y_{n+1}, \ldots, Y_m\}$ be an orthonormal basis of $\mathbb{R}^m$ such that $\{X_1, X_2, \ldots, X_n\}$ forms a basis of T_pM. We write

$$V = \sum_{i=1}^{n} v^i X_i + \sum_{i=n+1}^{m} v^i Y_i.$$

Then

$$\operatorname{div}^M V(p) = \sum_{i=1}^{n} \langle \bar{\nabla}_{X_i} V, X_i \rangle = \sum_{i=1}^{n} \big\langle \big(\bar{\nabla}_{X_i} V\big)^{\top}, X_i \big\rangle.$$

Thus for a smooth vector field $V \in \mathcal{T}(M)$, $\operatorname{div}^M V(p)$ is the trace of the linear map $T_pM \to T_pM,\ v \mapsto \nabla_v V$. In local coordinates,

$$\operatorname{div}^M V = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \big(\sqrt{g} v^i\big), \quad g = \det(g_{ij}).$$

Laplacian

The *Laplacian* of a C^2-function $f \in C^2(M)$ is defined as

$$\Delta^M f = \operatorname{div}^M \nabla^M f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\big(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j}\big).$$

In *normal coordinates* at p, we have the simple formula

$$\Delta^M f(p) = \sum_{i=1}^{n} \partial_i \partial_f f(p).$$

Jacobi formula

Lemma [Jacobi formula]

Let $a_{ij} \colon \mathbb{R}^d \to \mathbb{R}$ be smooth functions, with $i, j = 1, \ldots, n$, and let $A = (a_{ij})$. Then in the open set $\{x \in \mathbb{R}^m \colon \det A > 0\}$ we have

$$\frac{\partial}{\partial x^\ell} \log \det A = \operatorname{tr}\left(\frac{\partial A}{\partial x^\ell} A^{-1}\right)$$

for $\ell = 1, \ldots, d$.

Writing $A^{-1} = (a^{ij})$, the right hand side reads as

$$\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x^\ell} a^{ji},$$

and so

$$\frac{\partial \det A}{\partial x^\ell} = \det A \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x^\ell} a^{ji}. \tag{2}$$

Mean curvature and Laplacian

Suppose that $M \subset \mathbb{R}^m$ is a smooth n-dimensional submanifold and let $\varphi\colon U \to \Omega \subset R^n$ be a chart defined in an open set $U \subset M$. Furthermore, let $F = \varphi^{-1}\colon \Omega \to U$ be local parametrization. As before, F induces a frame $\{\frac{\partial F}{\partial x^j}\}$,

$$\left(\frac{\partial F}{\partial x^j}\right)_p = \left(\frac{\partial F_1}{\partial x^j}(\varphi(p)), \ldots, \frac{\partial F_m}{\partial x^j}(\varphi(p))\right) \in T_pM,$$

on U.
Now

$$\bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j} = \frac{\partial^2 F}{\partial x^i \partial x^j},$$

$$\left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j}\right)_p = \left(\frac{\partial^2 F_1}{\partial x^i \partial x^j}, \ldots, \frac{\partial^2 F_m}{\partial x^i \partial x^j}\right)(\varphi(p)) \in \mathbb{R}^m.$$

Mean curvature and Laplacian

Hence the mean curvature vector H_p at $p \in U$ is given by

$$
\begin{aligned}
H_p &= \sum_{i,j=1}^{n} g^{ij}(p)\, \mathrm{B}\big(\tfrac{\partial F}{\partial x^i}, \tfrac{\partial F}{\partial x^j}\big) \\
&= \sum_{i,j=1}^{n} g^{ij}(p) \left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j} \right)_p^{\perp} \\
&= \left(\sum_{i,j=1}^{n} g^{ij}(p) \frac{\partial^2 F}{\partial x^i \partial x^j} \big(\varphi(p)\big) \right)^{\perp}.
\end{aligned}
$$

Next we express the mean curvature vector as the Laplacian (on M) of the inclusion $\pi \colon M \hookrightarrow \mathbb{R}^m$.

Mean curvature and Laplacian

Theorem

Suppose that $M \subset \mathbb{R}^m$ is a smooth n-dimensional submanifold and let $\pi\colon M \hookrightarrow \mathbb{R}^m$, $\pi = (\pi_1, \dots, \pi_m)$, be the inclusion. Then

$$H_p = \Delta^M \pi(p) = \left(\Delta^M \pi_1, \dots, \Delta^M \pi_m\right)(p)$$

for $p \in M$.

Proof. Fix $p \in M$ and let $\varphi\colon U \to \Omega \subset \mathbb{R}^n$ be a chart at p and

$$\frac{\partial}{\partial x^i}, \quad i = 1, \dots, n,$$

the coordinate frame associated to the chart (U, φ). Furthermore, let $F = \varphi^{-1}\colon \Omega \to U$ be the corresponding (local) parametrization. Then, in fact,

$$\left(\frac{\partial}{\partial x^j}\right)_p \pi_i = \frac{\partial}{\partial x^j} \underbrace{\left(\pi_i \circ \varphi^{-1}\right)}_{=\pi_i \circ F = F_i} (\varphi(p)) = \frac{\partial F_i}{\partial x^j} (\varphi(p)).$$

Mean curvature and Laplacian

We claim that $\Delta^M \pi(p) \in N_p M$, that is

$$\Delta^M \pi(p) \cdot \tfrac{\partial F}{\partial x^k} = \Delta^M \pi(p) \cdot \tfrac{\partial \pi}{\partial x^k} = 0$$

for all $k = 1, \ldots, n$.

We compute by using the Jacobi formula and the symmetry of (g_{ij})

$$\begin{aligned}
\Delta^M \pi(p) \cdot \tfrac{\partial \pi}{\partial x^k} &= \Big(\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \Big(\sqrt{g} g^{ij} \tfrac{\partial \pi}{\partial x^j} \Big) \Big) \cdot \frac{\partial \pi}{\partial x^k} \\
&= \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \Big(\sqrt{g} g^{ij} \underbrace{\frac{\partial \pi}{\partial x^j} \cdot \frac{\partial \pi}{\partial x^k}}_{=g_{jk}} \Big) - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \Big(\sqrt{g} \underbrace{g^{ij} g_{jk}}_{\delta_{ik}} \Big) - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k}
\end{aligned}$$

Mean curvature and Laplacian

$$
\begin{aligned}
&= \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= \frac{1}{\sqrt{g}} \frac{1}{2\sqrt{g}} \frac{\partial g}{\partial x^k} - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} \frac{\partial}{\partial x^k} \left\langle \frac{\partial \pi}{\partial x^i}, \frac{\partial \pi}{\partial x^j} \right\rangle - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} \left(\frac{\partial^2 \pi}{\partial x^i \partial x^k} \cdot \frac{\partial \pi}{\partial x^j} + \frac{\partial^2 \pi}{\partial x^j \partial x^k} \cdot \frac{\partial \pi}{\partial x^i} \right) - \sum_{i,j=1}^{n} g^{ij} \frac{\partial \pi}{\partial x^j} \cdot \frac{\partial^2 \pi}{\partial x^i \partial x^k} \\
&= 0.
\end{aligned}
$$

Mean curvature and Laplacian

Thus $\Delta^M \pi(p) \in N_pM$ since $\left(\frac{\partial \pi}{\partial x^k}\right)_p$, $k = 1, \ldots, n$, forms a basis of T_pM. Furthermore,

$$\begin{aligned}\Delta^M \pi(p) &= \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial \pi}{\partial x^j}\right) \\ &= \underbrace{\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij}\right) \frac{\partial \pi}{\partial x^j}}_{\in T_pM} + \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 \pi}{\partial x^i \partial x^j}.\end{aligned}$$

On the other hand, since $\Delta^M \pi(p) \in N_pM$, we have

$$\begin{aligned}\Delta^M \pi(p) &= \left(\Delta^M \pi(p)\right)^{\perp} \\ &= \underbrace{\left(\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij}\right) \frac{\partial \pi}{\partial x^j}\right)^{\perp}}_{=0} + \left(\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 \pi}{\partial x^i \partial x^j}\right)^{\perp}\end{aligned}$$

Mean curvature and Laplacian

$$= \Big(\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 \pi}{\partial x^i \partial x^j} \Big)^{\perp} = H_p$$

as claimed.
We have proved:

$$\Delta^M \pi(p) = H_p.$$

First variation formula

Let $\Omega \subset \mathbb{R}^n$ be open, $f\colon \Omega \to \mathbb{R}^m$ an immersion, and $M = f\Omega$. Every $x \in \Omega$ has a neighborhood $U \subset \Omega$ such that $f|\Omega$ is an embedding.
Define the "tangent space" $T_{f(x)}M$ and the normal space $N_{f(x)}M$ as $T_{f(x)}M = T_{f(x)}U = df(x)\mathbb{R}^n$ and $N_{f(x)}M = N_{f(x)}U$.
Let $\varphi \in C_0^\infty(\Omega)$ be a real-valued function and let $N\colon \Omega \to \mathbb{S}^{m-1}$ be smooth such that $N_x = N(x) \in N_{f(x)}M \; \forall x \in \Omega$.
Define a variation of M (more precisely, a variation of the immersion $f\colon \Omega \to \mathbb{R}^m$) with compact support as

$$F\colon \Omega \times]-\varepsilon, \varepsilon[\to \mathbb{R}^m, \quad F(x,t) = f(x) + t\varphi(x)N_x,$$

with $\varepsilon > 0$ small enough.
Let $\{\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}, \partial_t\}$ be the standard basis of $\mathbb{R}^{n+1}$ and define vector fields F_{x_i} and F_t along F by setting

First variation formula

$$F_{x_i}(x,t) = dF(x,t)\partial_{x_i} \quad \text{and} \quad F_t(x,t) = dF(x,t)\partial_t.$$

Then F_{x_i} and F_t commute because

$$[F_{x_i}, F_t] = dF \underbrace{[\partial_{x_i}, \partial_t]}_{=0} = 0.$$

Note that $F_t(x,0) = dF(x,0)\partial_t = \varphi(x)N_x \in N_{f(x)}M$.
Define

$$g_{ij}(x,t) = \langle F_{x_i}(x,t), F_{x_j}(x,t)\rangle \quad \text{and} \quad g(x,t) = \det g_{ij}(x,t).$$

Then the volume of $M_t = F(\Omega, t)$ is

$$\operatorname{Vol} M_t = \int_\Omega \sqrt{g(x,t)}\,dx.$$

First variation formula

Hence

$$\frac{d}{dt}\mathrm{Vol}\, M_{t|t=0} = \int_\Omega \frac{\partial}{\partial t}\sqrt{g(x,t)}_{|t=0} dx$$

$$= \frac{1}{2}\int_\Omega \frac{1}{\sqrt{g(x,0)}} \frac{\partial}{\partial t} g(x,t)_{|t=0} dx$$

$$= \frac{1}{2}\int_\Omega \sqrt{g(x,0)} \sum_{i,j=1}^{n} g^{ij}(x,0) \frac{\partial g_{ij}(x,t)}{\partial t}_{|t=0} dx$$

$$= \frac{1}{2}\int_\Omega \sqrt{g(x,0)} \sum_{i,j=1}^{n} g^{ij}(x,0) \frac{\partial}{\partial t}\langle F_{x_i}, F_{x_j}\rangle(x,t)_{|t=0} dx$$

$$= \frac{1}{2}\int_\Omega \sqrt{g(x,0)} \sum_{i,j=1}^{n} g^{ij}(x,0) \left(\langle \bar\nabla_{F_t} F_{x_i}, F_{x_j}\rangle + \langle \bar\nabla_{F_t} F_{x_j}, F_{x_i}\rangle\right)(x,0) dx$$

$$= \int_\Omega \sqrt{g(x,0)} \sum_{i,j=1}^{n} g^{ij}(x,0) \langle \bar\nabla_{F_t} F_{x_i}, F_{x_j}\rangle(x,0) dx$$

First variation formula

$$
\begin{aligned}
&= \int_\Omega \sqrt{g(x,0)} \sum_{i,j=1}^{n} g^{ij}(x,0) \langle \bar{\nabla}_{F_{x_i}} F_t, F_{x_j} \rangle (x,0) dx \\
&= \int_\Omega \sqrt{g(x,0)} \sum_{i,j=1}^{n} g^{ij}(x,0) \langle \bar{\nabla}_{F_{x_i}} (\varphi(x) N_x), F_{x_j} \rangle (x,0) dx \\
&= - \int_\Omega \sqrt{g(x,0)} \sum_{i,j=1}^{n} g^{ij}(x,0) \langle \mathrm{B}\big(F_{x_i}, F_{x_j}\big), \varphi(x) N_x \rangle (x,0) dx \\
&= - \int_\Omega \sqrt{g(x,0)} \langle H_{f(x)}, \varphi(x) N_x \rangle (x,0) dx \\
&=: - \int_M \langle H, V \rangle .
\end{aligned}
$$

First variation formula

Above $H_{f(x)}$ denotes the mean curvature vector at $f(x)$ of fU, with $f|U$ an embedding. Moreover,

$$-\int_M \langle H, V \rangle$$

is a shorthand notation in case the immersion $f \colon \Omega \to \mathbb{R}^m$ is non-injective, whereas $V_p = \varphi(f^{-1}(p))N_{f^{-1}(p)}$ for an injective immersion f. *Conclusion:* If $H \equiv 0$, then $M = M_0$ is a critical point for the volume functional. Otherwise, "deforming" M into the direction of H_p decreases the volume.

Riemannian manifold

Let $\tilde{M}$ be an m-dimensional C^∞-manifold, $T_x\tilde{M}$ the tangent space at $x \in \tilde{M}$, and

$$T\tilde{M} = \bigsqcup_{x \in \tilde{M}} T_x\tilde{M}$$

the tangent bundle. [Note: $T\tilde{M}$ is a $2m$-dimensional smooth manifold.]
A *Riemannian metric (tensor)* on $\tilde{M}$ is a 2-covariant tensor field $\tilde{g} \in \mathcal{T}^2(\tilde{M})$ that is symmetric (i.e. $\tilde{g}(X, Y) = \tilde{g}(Y, X)$) and positive definite (i.e. $\tilde{g}(X_x, X_x) > 0$ if $X_x \neq 0$). A smooth manifold $\tilde{M}$ with a given Riemannian metric $\tilde{g}$ is called a *Riemannian manifold* $(\tilde{M}, \tilde{g})$.
A Riemannian metric thus defines an inner product on each $T_x\tilde{M}$, written as $\langle v, w \rangle = \langle v, w \rangle_x = \tilde{g}(v, w)$ for $v, w \in T_x\tilde{M}$.
The inner product varies smoothly in x in the sense that for every $X, Y \in \mathcal{T}(\tilde{M})$, the function $\tilde{M} \to \mathbb{R}$, $x \mapsto \tilde{g}(X_x, Y_x)$, is C^∞.

Riemannian connection

Remark

Given a Riemannian manifold $(\tilde{M}, \tilde{g})$, there exists a *unique* mapping

$$\tilde{\nabla} \colon \mathcal{T}(\tilde{M}) \times \mathcal{T}(\tilde{M}) \to \mathcal{T}(\tilde{M}), \quad \tilde{\nabla}(X, Y) = \tilde{\nabla}_X Y,$$

called the *Riemannian* (or the *Levi-Civita*) *connection* on $(\tilde{M}, \tilde{g})$ satisfying the properties 1.-5. below.

1. $\tilde{\nabla}_X Y$ is C^∞-linear in X: for every functions $f, g \in C^\infty(\tilde{M})$ and vector fields $X, Y, V \in \mathcal{T}(\tilde{M})$

$$\tilde{\nabla}_{fX+gY} V = f\tilde{\nabla}_X V + g\tilde{\nabla}_Y V;$$

2. $\tilde{\nabla}_X Y$ is $\mathbb{R}$-linear in Y: for every $a, b \in \mathbb{R}, X, Y, V \in \mathcal{T}(\tilde{M})$

$$\tilde{\nabla}_X(aY + bV) = a\tilde{\nabla}_X Y + b\tilde{\nabla}_X V;$$

Riemannian connection

3. $\tilde{\nabla}$ satisfies the Leibniz rule: for every $f \in C^\infty(\tilde{M}), X, Y \in \mathcal{T}(\tilde{M})$

$$\tilde{\nabla}_X(fY) = f\tilde{\nabla}_X Y + (Xf)Y;$$

4. $\tilde{\nabla}$ is *torsion-free*: for every $X, Y \in \mathcal{T}(\tilde{M})$

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X = [X, Y];$$

5. $\tilde{\nabla}$ is *compatible* with the Riemannian metric $\langle \cdot, \cdot \rangle$ of $\tilde{M}$: for every $X, Y, Z \in \mathcal{T}(\tilde{M})$

$$X\langle Y, Z\rangle = \langle \tilde{\nabla}_X Y, Z\rangle + \langle Y, \tilde{\nabla}_X Z\rangle.$$

Riemannian curvature tensor and sectional curvature

The Riemannian curvature on $\tilde{M}$ is the tensor field

$$\tilde{R}\colon \mathcal{T}(\tilde{M}) \times \mathcal{T}(\tilde{M}) \times \mathcal{T}(\tilde{M}) \times \to \mathcal{T}(\tilde{M})$$

defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z.$$

The *sectional curvature* of a 2-dimensional subspace $P \subset T_p\tilde{M}$ spanned by vectors $v, w \in T_p\tilde{M}$ is defined by

$$\tilde{K}(P) = \frac{\langle \tilde{R}(v,w)w, v\rangle}{|v \wedge w|^2},$$

where

$$|v \wedge w| = \sqrt{|v|^2|w|^2 - \langle v, w\rangle^2}$$

is the area of the parallelogram spanned by v and w.

Ricci curvature

The Ricci curvature on $\tilde{M}$ is the tensor field defined by

$\widetilde{\mathrm{Ric}}(x,y) = \mathrm{tr}\,(z \mapsto R(z,x)y) =$ the trace of the linear map $z \mapsto R(z,x)y$.

Hence if $e_1, \ldots, e_m$ is an orthonormal basis of $T_p\tilde{M}$, then

$$\widetilde{\mathrm{Ric}}(x,y) = \sum_{i=1}^{m} \langle \tilde{R}(e_i,x)y, e_i \rangle = \sum_{i=1}^{m} \langle \tilde{R}(x,e_i)e_i, y \rangle.$$

We set $\widetilde{\mathrm{Ric}}(x) = \widetilde{\mathrm{Ric}}(x,x)$. If $|x| = 1$, $\widetilde{\mathrm{Ric}}(x)$ is called the *Ricci curvature in the direction* x.
Hence if $|x| = 1$ and $e_1, \ldots, e_{m-1} \in T_p\tilde{M}$ such that $x, e_1, \ldots, e_{m-1}$ is an orthonormal basis of $T_p\tilde{M}$, we get

$$\widetilde{\mathrm{Ric}}(x) = \underbrace{\langle \tilde{R}(x,x)x, x \rangle}_{=0} + \sum_{i=1}^{m-1} \langle \tilde{R}(x,e_i)e_i, x \rangle = \sum_{i=1}^{m-1} \tilde{K}(P_i),$$

where $P_i \subset T_p\tilde{M}$ is the plane spanned by x and e_i.

Riemannian structure of submanifolds of $\tilde{M}$.

Let M be an n-dimensional smooth submanifold of $(\tilde{M}, \tilde{g})$. Then $\tilde{g}$ induces a Riemannian metric g on M: for every $p \in M$ and for every vectors $v, w \in T_pM$,

$$g(v, w) = \langle v, w \rangle = \tilde{g}(v, w).$$

The Riemannian connection ∇,

$$(\nabla_X Y)_p = (\tilde{\nabla}_{\tilde{X}} \tilde{Y})_p^\top,$$

the second fundamental form B,

$$\mathrm{B}(X, Y) = \left(\tilde{\nabla}_{\tilde{X}} \tilde{Y}\right)^\perp,$$

the mean curvature vector

$$H_p = \sum_{i=1}^n \mathrm{B}(X_i, X_i),$$

with $X_1, \ldots, X_n$ an orthonormal basis of T_pM,

Riemannian structure of submanifolds of $\tilde{M}$.

and the Riemannian curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

are defined as in the case of submanifolds of $\mathbb{R}^m$.
As in the Euclidean setting, we define:

Definition

A submanifold $M \subset \tilde{M}$ is *minimal* if $H \equiv 0$ on M.

Minimal graph equation

Let M be an n-dimensional Riemannian manifold, $\Omega \subset M$ a bounded open set, and $u\colon \bar{\Omega} \to \mathbb{R}$ a C^2-function. The graph of u,

$$\Sigma = \{(x, u(x))\colon x \in \bar{\Omega}\} \subset M \times \mathbb{R} := \tilde{M},$$

is an n-dimensional (C^2-smooth) submanifold of $M \times \mathbb{R}$. [Note: $\tilde{M} = M \times \mathbb{R}$ equipped with the product structure.]
Its (n-dim. measure) volume is given by

$$\mathcal{A}(\Sigma) = \int_\Omega \sqrt{1 + |\nabla u|^2} dV.$$

Here ∇u is the *gradient* of u is defined by

$$\langle \nabla u, X \rangle = Xu$$

for all vector fields X. Thus

$$\nabla u = \sum_{i=1}^{n} (X_i u) X_i$$

if $X_1, \ldots, X_n$ are orthonormal.

Minimal graph equation

As in the Euclidean case, a function $u \in C^2(\Omega)$ is a critical point of the area (or volume) functional if

$$\mathcal{M}[u] := \text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{3}$$

Here div is the *divergence* defined by

$$\text{div}\, X = \text{tr}\,(\xi \mapsto \nabla_\xi X)$$

for C^1-smooth vector fields X. Again, if u is a solution of (3) in Ω, its graph Σ_u is a minimal submanifold of $\tilde{M} = M \times \mathbb{R}$.
Furthermore, the function ("height function") $\Sigma_u \to \mathbb{R}$,

$$(x, u(x)) \mapsto u(x),$$

is a harmonic function on Σ_u and the mapping ("vertical projection") $\Sigma_u \to M$,

$$(x, u(x)) \mapsto x,$$

is a harmonic mapping on Σ_u.

Dirichlet problem

Next I will explain the idea of a proof of the following theorem:

Theorem

Suppose that $\Omega \Subset M$ is a smooth relatively compact open set whose boundary has positive mean curvature with respect to inwards pointing unit normal field. Then for each $\psi \in C^{2,\alpha}(\bar{\Omega})$ there exists a unique $u \in C^{\infty}(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$ that solves the minimal graph equation (3) in Ω with boundary values $u|\partial\Omega = \psi|\partial\Omega$.

Since $\partial\Omega \subset M$ is a hypersurface (co-dimension 1), "positive mean curvature with respect to inwards pointing unit normal field" just means that the mean curvature vector $H_p \neq 0$ of $\partial\Omega$ is parallel to the inwards pointing unit normal vector at every $p \in \partial\Omega$.

(Nonlinear) continuity method

Jürgen Jost: Partial Differential Equations:
"Connect what you want to know to what you know already.
This is the continuity method. The idea is that, if you can connect your given problem continuously with another, simpler, problem that you can already solve, then you can also solve the former. Of course, the continuation of solutions requires careful control."
Let $\psi \in C^{2,\alpha}(\bar{\Omega})$ be given. Denote

$$A = \{t \in [0,1] \colon \exists u_t \in C^{2,\alpha}(\bar{\Omega}) \text{ such that } \mathcal{M}[u_t] = 0 \text{ in } \Omega \text{ and } u_t|\partial\Omega = t\psi\}.$$

The idea is simple:
Prove that $A \neq \emptyset$ is both open and closed in $[0,1]$, hence $A = [0,1]$ and, in particular, there exists a solution u, with $u|\partial\Omega = \psi|\partial\Omega$.

1. $A \neq \emptyset$ since $0 \in A$. (The constant function $u_0 \equiv 0$ is a solution.)
2. A is open. This is a consequence of the implicit function theorem.
3. A is closed. This follows from *a priori* estimates for (smooth) solutions together with Schauder estimates.

Implicit function theorem

Recall:

Let E, F be Banach spaces, $U \subset E$ open, and $x_0 \in U$. A function $f\colon U \to F$ is (Fréchet) differentiable at x_0 if there exists $A \in L(E,F)$ (= continuous linear), called the differential of f at x_0, such that

$$f(x_0 + h) = f(x_0) + Ah + o(h) \quad \text{as } h \to 0.$$

Implicit function theorem

Let E, F, G be Banach spaces, $\Omega \subset E \times F$ open, $f \in C^1(\Omega, G)$, and $(x_0, y_0) \in \Omega$, with $f(x_0, y_0) = 0$. Let $D_2 f(x_0, y_0) \in L(F, G)$ be the differential at x_0 of the map $y \mapsto f(x_0, y)$. If $D_2 f(x_0, y_0)\colon F \to G$ is a linear isomorphism, then there exist neighborhoods $U \ni x_0,\ V \ni y_0$, and a differentiable map $g\colon U \to V$ such that $f\big(x, g(x)\big) = 0$ and $f(x,y) = 0$ if and only if $y = g(x)$, for all $(x,y) \in U \times V$.

Implicit function theorem

Let $\Omega \Subset M$ be a relatively compact open set. Denote

$$[u]_{\alpha;\Omega} = \sup\left\{\frac{|u(x)-u(y)|}{d(x,y)^\alpha} : x,y\in\Omega, x\neq y\right\},\ 0<\alpha\le 1,$$

$$|D^k u|_{0;\Omega} = \sup_{|\beta|=k}\sup_{\Omega}|D^\beta u|,\ k=0,1,2,\ldots,$$

$$[D^k u]_{\alpha;\Omega} = \sup_{|\beta|=k}[D^\beta u]_{\alpha;\Omega},$$

$$\|u\|_{C^k(\bar\Omega)} = \sum_{j=0}^{k}|D^j u|_{0;\Omega},$$

$$\|u\|_{C^{k,\alpha}(\bar\Omega)} = \|u\|_{C^k(\bar\Omega)} + [D^k u]_{\alpha;\Omega}.$$

The *Hölder spaces* $C_0^{k,\alpha}(\bar\Omega)\subset C^{k,\alpha}(\bar\Omega)\subset C^k(\bar\Omega),\ k=0,1,2,\ldots,$ are Banach spaces equipped with norms $\|\cdot\|_{C^{k,\alpha}(\bar\Omega)}$.

A is open

To prove that the set

$A = \{t \in [0,1] \colon \exists u_t \in C^{2,\alpha}(\bar{\Omega})$ such that $\mathcal{M}[u_t] = 0$ in Ω and $u_t|\partial\Omega = t\psi\}$.

is open in $[0,1]$, let $t_0 \in A$. Need to show that $(t_0 - \varepsilon, t_0 + \varepsilon) \cap [0,1] \subset A$ for some $\varepsilon > 0$.

We apply the implicit function theorem to the mapping $f \colon \mathbb{R} \times C_0^{2,\alpha}(\bar{\Omega}) \to C^{\alpha}(\bar{\Omega})$,

$$f(t,u) = \mathcal{M}[u + t\psi] = \operatorname{div} \frac{\nabla(u + t\psi)}{\sqrt{1 + |\nabla(u + t\psi)|^2}}.$$

Note that $t \in A$ if and only if $f(t, v_t) = 0$ for some $v_t \in C_0^{2,\alpha}(\bar{\Omega})$ since

$$(v_t + t\psi)|\partial\Omega = t\psi|\partial\Omega \quad \text{and} \quad \mathcal{M}[v_t + t\psi] = f(t, v_t) = 0,$$

and so $u_t = v_t + t\psi$ is the desired solution.

A is open

Thus let $(t_0, v_0) \in A \times C_0^{2,\alpha}(\bar{\Omega})$. Then $f(t_0, v_0) = 0$. Furthermore, f is C^1 and $D_2 f(t_0, v_0) \colon C_0^{2,\alpha}(\bar{\Omega}) \to C^{\alpha}(\bar{\Omega})$ is a linear isomorphism by the theory of uniformly elliptic linear operators (maximum principles, Schauder estimates, existence and regularity of solutions to Dirichlet problem; see e.g. Gilbarg-Trudinger).

The implicit function theorem then implies that A is open in $[0, 1]$.

A is closed

To prove that A is closed, let $t_i \in A$, with $t_i \to t \in [0,1]$. Need to show that $t \in A$.
Let $u_i \in C^{2,\alpha}(\bar{\Omega})$ be the solution $\mathcal{M}[u_i] = 0$, with $u_i|\partial\Omega = t_i\psi|\partial\Omega$.
It suffices to show that

there exists a subsequence (u_i) such that $u_i \to u \in C^{2,\alpha}(\bar{\Omega})$ in $C^2(\bar{\Omega})$ norm

since then

$$u|\partial\Omega = \lim_{i\to\infty} u_i|\partial\Omega = \lim_{i\to\infty} t_i\psi|\partial\Omega = t\psi|\partial\Omega$$

and

$$\mathcal{M}[u] = f(t, u - t\psi) = f\left(\lim_{i\to\infty}(t_i, u_i - t_i\psi)\right)$$
$$\lim_{i\to\infty} f(t_i, u_i - t_i\psi) = \lim_{i\to\infty} \mathcal{M}[u_i] = 0.$$

A is closed

The existence of such a subsequence follows from *a priori* estimates

$$\sup_{\Omega} |u_i| \leq c \quad \text{and} \quad \sup_{\Omega} |\nabla u_i| \leq c,$$

and Schauder estimates

$$\|u_i\|_{C^{2,\gamma}(\bar{\Omega})} \leq c,$$

with constant $c < \infty$ independent of i.
The estimate

$$\sup_{\Omega} |u_i| \leq c$$

follows from the maximum principle (ψ is a bounded function and constant functions are solutions).
Next we discuss about (interior and boundary) gradient estimates

$$\sup_{\Omega} |\nabla u_i| \leq c.$$

Boundary gradient estimate: Idea

Suppose that $\Omega \Subset M$ is a smooth relatively compact open set whose boundary has positive mean curvature with respect to inwards pointing unit normal field. We say that Ω is strictly *mean convex*.
Let $\psi \in C^{2,\alpha}(\bar{\Omega})$ and consider functions $w^+, w^- \colon \bar{\Omega} \to \mathbb{R}$,

$$w^+(x) = t_i\psi(x) + \varphi\big(d(x)\big) \quad \text{and} \quad w^-(x) = t_i\psi(x) - \varphi\big(d(x)\big),$$

where $t_i \in A$, $d(x) = \operatorname{dist}(x, \partial\Omega) = \min\{d(x,y) \colon y \in \partial\Omega\}$, and

$$\varphi(s) = c_1 \log(1 + c_2 s). \tag{4}$$

Denote

$$\Omega_s = \{x \in \Omega \colon d(x) < s\} \quad \text{and} \quad \Gamma_s = \{x \in \Omega \colon d(x) = s\}.$$

If $x \in \Gamma_s$, for $s \leq s_0$ small enough, $-\Delta d(x)$ is the sum of the principal curvatures of Γ_s with respect to inwards pointing unit normal.
Since Ω is strictly mean convex, we conclude that $\Delta d(x) \leq 0$ for $x \in \Omega_s$ for $s \leq s_0$ small enough.

Boundary gradient estimate: Idea

By choosing constants c_1, c_2 in (4) properly, we conclude that w^+ is a *supersolution* and w^- is a *subsolution*. Furthermore, $w^\pm|\partial\Omega = u_i|\partial\Omega$, and

$$w^+ \geq \sup_{\partial\Omega} u_i, \quad w^- \leq \inf_{\partial\Omega} u_i \quad \text{on } \Gamma_{t_0}.$$

It follows that

$$\sup_{\partial\Omega} |\nabla u_i| \leq \max\left\{\sup_{\partial\Omega} |\nabla w^+|, \sup_{\partial\Omega} |\nabla w^-|\right\} \leq c < \infty$$

for the solution $u_i \in C^{2,\alpha}(\bar{\Omega})$, with $u_i|\partial\Omega = t_i\psi|\partial\Omega$.

Interior gradient estimates

I will sketch the proof(s) of the following gradient estimate(s):

Lemma [Rosenberg-Schulze-Spruck]

Let $\Omega \Subset M$ be a relatively compact open set and let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution to the minimal graph equation $\mathcal{M}[u] = 0$ in Ω. Then

$$\sup_{\Omega} \sqrt{1 + |\nabla u|^2} \leq \sup_{\Omega} e^{-\alpha u} \cdot \sup_{\partial\Omega} \left(e^{\alpha u} \sqrt{1 + |\nabla u|^2} \right),$$

where

$$\alpha^2 = \sup\{\max\{-\operatorname{Ric}(\gamma,\gamma), 0\} \colon \gamma \in T_pM,\ |\gamma| = 1,\ p \in \Omega\}.$$

Interior gradient estimates

For the next lemma, suppose that $\Omega \Subset M$ is a relatively compact open set, $x \in \Omega$, and $B(x,\rho) \subset \Omega$, where $\rho < \mathrm{inj}(x)$, the injectivity radius of M at x.

Lemma [Spruck]

Let $u \in C^3(\Omega)$ be a non-negative solution of the mean curvature equation

$$\operatorname{div} \frac{\nabla u(y)}{\sqrt{1+|\nabla u(y)|^2}} = H(y)$$

in Ω. Then

$$\sqrt{1+|\nabla u(x)|^2} \le 32 \max\left\{1, \left(u(x)/\rho\right)^2\right\} e^{16Cu(x)} e^{16(u(x)/\rho)^2}$$

for a constant C independent of u, but depending on the C^1-norm of H, on a lower bound for the sectional curvatures of M, and on an upper bound for Δd^2 on Ω.

"Bochner-type" formula

Both lemmas are proved by applying maximum principle to a subsolution of an elliptic PDE. For that purpose, we need the following "Bochner-type" formula.

Theorem

Let M^m be a Riemannian manifold and let $N = M^m \times \mathbb{R}$ be equipped with the product structure. Let E_{m+1} be the unit vector field such that

$$E_{m+1}(p,t) = \frac{\partial}{\partial t} \quad \forall (p,t) \in N.$$

Let $\Sigma \subset N$ be an m-dimensional (smooth) hypersurface with induced structure, η a smooth unit normal vector field to Σ, and define $f(x) = \langle \eta_x, E_{m+1}(x) \rangle$ for $x \in \Sigma$. Then

$$\Delta f = \Delta^{\Sigma} f = -\langle E_{m+1}^{\top}, \nabla h \rangle - \left(\widetilde{\mathrm{Ric}}(\eta) + \|\mathrm{B}\|^2 \right) f,$$

"Bochner-type" formula

where $h = \langle H, \eta \rangle$ is the scalar mean curvature of Σ (w.r.t. η), $\nabla h = \nabla^{\Sigma} h$ its gradient, $\|\mathrm{B}\|^2$ the squared norm of B, and $\widetilde{\mathrm{Ric}}$ the Ricci curvature on N.

Remarks

1. $$\|\mathrm{B}\|^2 = \sum_{i,j=1}^{m} |\mathrm{B}(E_i, E_j)|^2 = \sum_{i=1}^{m} \kappa_i^2,$$

   where $E_1, \ldots, E_m$ is a (local) orthonormal frame on Σ and κ_i's are the principal curvatures.
2. $f(x) = \langle \eta_x, E_{m+1}(x) \rangle$ is the "vertical ($\mathbb{R}$-)component" of η_x.

"Bochner-type" formula

Corollary

Suppose that Σ has a constant mean curvature. Then

$$\Delta f = -\left(\widetilde{\mathrm{Ric}}(\eta) + \|\mathrm{B}\|^2\right) f.$$

Remark

If Σ is the graph of a solution $u\colon \Omega \to \mathbb{R}$ of the minimal graph equation in $\Omega \subset M$, then $h \equiv 0$ and

$$f = \frac{1}{\sqrt{1 + |\nabla^M u|^2}}.$$

Proof of the "Bochner-type" formula

Fix $x \in \Sigma$ and let $E_1(x), \dots, E_m(x)$ be an orthonormal basis of $T_x\Sigma$ consisting of the eigenvectors of the Weingarten map $L\colon T_x\Sigma \to T_x\Sigma$, with eigenvalues κ_i. Extend $E_1(x), \dots, E_m(x)$ to a geodesic frame $E_1, \dots, E_m$ in a neighborhood of x in Σ (thus $(\nabla_{E_j} E_i)_x = 0$).
Then

$$\Delta f(x) = \sum_{i=1}^{m} E_i E_i f(x).$$

We compute at x:

$$\begin{aligned}
\Delta f(x) &= \sum_{i=1}^{m} E_i E_i f = \sum_i E_i E_i \langle \eta, E_{m+1} \rangle \\
&= \sum_i E_i \big(\langle \bar\nabla_{E_i} \eta, E_{m+1} \rangle + \langle \eta, \underbrace{\bar\nabla_{E_i} E_{m+1}}_{=0} \rangle \big) \\
&= \sum_i \langle \bar\nabla_{E_i} \bar\nabla_{E_i} \eta, E_{m+1} \rangle.
\end{aligned}$$

Proof of the "Bochner-type" formula

Write

$$E_{m+1} = \sum_{j=1}^{m} e_j E_j + f\eta, \quad f = \langle \eta, E_{m+1} \rangle.$$

Then

$$\begin{aligned} \Delta f(x) &= \sum_i \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} \eta, E_{m+1} \rangle \\ &= \sum_{i,j} e_j \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} \eta, E_j \rangle + \sum_i f \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} \eta, \eta \rangle. \end{aligned}$$

We have at x:

$$\langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} \eta, E_j \rangle = \langle \bar{R}(E_j, E_i) E_i, \eta \rangle - E_j \langle \bar{\nabla}_{E_i} E_i, \eta \rangle \tag{5}$$

and

Proof of the "Bochner-type" formula

$$
\begin{aligned}
\langle \bar\nabla_{E_i}\bar\nabla_{E_i}\eta,\eta\rangle &= -\langle \bar\nabla_{E_i}\underbrace{(LE_i)}_{\in T\Sigma},\eta\rangle \\
&= -\underbrace{\langle \underbrace{\nabla_{E_i}(LE_i)}_{\in T\Sigma},\eta\rangle}_{=0} - \langle \big(\bar\nabla_{E_i}(LE_i)\big)^{\perp},\eta\rangle \\
&= -\langle \mathrm{B}(E_i,LE_i),\eta\rangle = \langle \underbrace{\bar\nabla_{E_i}\eta}_{=-LE_i},LE_i\rangle \\
&= -\langle LE_i,LE_i\rangle = -\kappa_i^2.
\end{aligned}
$$

So,

$$
\begin{aligned}
\Delta f(x) &= \sum_i \langle \bar\nabla_{E_i}\bar\nabla_{E_i}\eta, E_{m+1}\rangle \\
&= \sum_{i,j} e_j\left(\langle \bar R(E_j,E_i)E_i,\eta\rangle - E_j\langle \bar\nabla_{E_i}E_i,\eta\rangle\right) - f\sum_i \kappa_i^2
\end{aligned}
$$

Proof of the "Bochner-type" formula

$$= \sum_i \left(\langle \bar{R}(E_{m+1} - f\eta, E_i)E_i, \eta \rangle - \left(\sum_j e_j E_j \right) \langle \bar{\nabla}_{E_i} E_i, \eta \rangle \right) - f\|\mathrm{B}\|^2$$

$$= \underbrace{\sum_i \langle \bar{R}(E_{m+1}, E_i)E_i, \eta \rangle}_{=\widetilde{\mathrm{Ric}}(E_{m+1},\eta)=0} - f \underbrace{\sum_i \langle \bar{R}(\eta, E_i)E_i, \eta \rangle}_{=\widetilde{\mathrm{Ric}}(\eta)}$$

$$- \sum_i \left(\sum_j e_j E_j \right) \langle \bar{\nabla}_{E_i} E_i, \eta \rangle - f\|\mathrm{B}\|^2$$

$$= -f\widetilde{\mathrm{Ric}}(\eta) - \left(\sum_j e_j E_j \right) \sum_i \Big(\underbrace{\langle \underbrace{\nabla_{E_i} E_i}_{\in T\Sigma}, \eta \rangle}_{=0} + \langle \underbrace{\big(\bar{\nabla}_{E_i} E_i\big)^{\perp}}_{=\mathrm{B}(E_i,E_i)}, \eta \rangle \Big) - f\|\mathrm{B}\|^2$$

Proof of the "Bochner-type" formula

$$= -f\widetilde{\mathrm{Ric}}(\eta) - \underbrace{\Big(\sum_j e_j E_j\Big)}_{=E_{m+1}^{\top}} \underbrace{\langle H, \eta \rangle}_{=h} - f\|\mathrm{B}\|^2$$

$$= -f\widetilde{\mathrm{Ric}}(\eta) - \langle E_{m+1}^{\top}, \nabla h \rangle - f\|\mathrm{B}\|^2.$$

We are left with the *Proof of* (5):

$$\langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} \eta, E_j \rangle = \langle \bar{R}(E_j, E_i)E_i, \eta \rangle - E_j \langle \bar{\nabla}_{E_i} E_i, \eta \rangle.$$

Proof of the "Bochner-type" formula

First we note that

$$
\begin{aligned}
E_j\langle\bar\nabla_{E_i}E_i,\eta\rangle &= \langle\bar\nabla_{E_j}\bar\nabla_{E_i}E_i,\eta\rangle + \langle\bar\nabla_{E_i}E_i,\underbrace{\bar\nabla_{E_j}\eta}_{\in T\Sigma}\rangle \\
&= \langle\bar\nabla_{E_j}\bar\nabla_{E_i}E_i,\eta\rangle + \langle\underbrace{\nabla_{E_i}E_i}_{=0 \text{ at } x},\bar\nabla_{E_j}\eta\rangle \\
&= \langle\bar\nabla_{E_j}\bar\nabla_{E_i}E_i,\eta\rangle.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\langle\bar\nabla_{E_i}\bar\nabla_{E_i}\eta,E_j\rangle = \langle\bar R(E_j,E_i)E_i,\eta\rangle - E_j\langle\bar\nabla_{E_i}E_i,\eta\rangle \\
\iff\ &\langle\bar\nabla_{E_i}\bar\nabla_{E_i}\eta,E_j\rangle = \langle\bar\nabla_{E_j}\bar\nabla_{E_i}E_i,\eta\rangle - \langle\bar\nabla_{E_i}\bar\nabla_{E_j}E_i,\eta\rangle \\
&\qquad - \langle\bar\nabla_{\underbrace{[E_j,E_i]}_{=0 \text{ at } x}}E_i,\eta\rangle - \langle\bar\nabla_{E_j}\bar\nabla_{E_i}E_i,\eta\rangle - \overbrace{\langle\bar\nabla_{E_i}E_i,\underbrace{\bar\nabla_{E_j}\eta}_{\in T\Sigma}\rangle}^{}_{=\langle\underbrace{\nabla_{E_i}E_i}_{=0},\bar\nabla_{E_j}\eta\rangle=0}
\end{aligned}
$$

Proof of the "Bochner-type" formula

$$\iff \langle \bar\nabla_{E_i}\bar\nabla_{E_i}\eta, E_j\rangle = -\langle \bar\nabla_{E_i}\bar\nabla_{E_j}E_i, \eta\rangle \tag{6}$$

Remains to verify (6).
Since $\langle E_i, \eta\rangle \equiv 0$ (along Σ), we have $E_j\langle E_i, \eta\rangle = 0$.
Hence

$$\begin{aligned} 0 &\equiv \langle \bar\nabla_{E_j}E_i, \eta\rangle + \langle E_i, \bar\nabla_{E_j}\eta\rangle \\ &= \langle \bar\nabla_{E_j}E_i, \eta\rangle + \underbrace{\langle E_i, -LE_j\rangle}_{\langle -LE_i, E_j\rangle} \\ &= \langle \bar\nabla_{E_j}E_i, \eta\rangle + \langle E_j, \bar\nabla_{E_i}\eta\rangle \end{aligned}$$

along Σ.
Therefore

$$\begin{aligned} 0 &= E_i\left(\langle \bar\nabla_{E_j}E_i, \eta\rangle + \langle E_j, \bar\nabla_{E_i}\eta\rangle\right) \\ &= \langle \bar\nabla_{E_i}\bar\nabla_{E_j}E_i, \eta\rangle + \langle \bar\nabla_{E_j}E_i, \bar\nabla_{E_i}\eta\rangle \end{aligned}$$

Proof of the "Bochner-type" formula

$$= \langle \bar\nabla_{E_i}\bar\nabla_{E_j}E_i, \eta\rangle + \langle \underbrace{\nabla_{E_j}E_i}_{=0 \text{ at } x} + \left(\bar\nabla_{E_j}E_i\right)^\perp, \underbrace{-LE_i}_{\in T\Sigma}\rangle$$

$$+ \langle \underbrace{\nabla_{E_i}E_j}_{=0 \text{ at } x} + \left(\bar\nabla_{E_i}E_j\right)^\perp, \underbrace{-LE_i}_{\in T\Sigma}\rangle + \langle E_j, \bar\nabla_{E_i}\bar\nabla_{E_i}\eta\rangle$$

$$= \langle \bar\nabla_{E_i}\bar\nabla_{E_j}E_i, \eta\rangle + \langle E_j, \bar\nabla_{E_i}\bar\nabla_{E_i}\eta\rangle$$

So, we have proved:

$$\Delta^\Sigma f = -\langle E_{m+1}^\top, \nabla h\rangle - \left(\widetilde{\mathrm{Ric}}(\eta) + \|\mathrm{B}\|^2\right) f,$$

where $f = \langle \eta, E_{m+1}\rangle$.

Proof of Rosenberg-Schulze-Spruck estimate

Let's recall:

Lemma [Rosenberg-Schulze-Spruck]

Let $\Omega \Subset M$ be a relatively compact open set and let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution to the minimal graph equation $\mathcal{M}[u] = 0$ in Ω. Then

$$\sup_{\Omega} \sqrt{1 + |\nabla u|^2} \le \sup_{\Omega} e^{-\alpha u} \cdot \sup_{\partial\Omega} \left(e^{\alpha u} \sqrt{1 + |\nabla u|^2} \right), \qquad (7)$$

where

$$\alpha^2 = \sup\{\max\{-\operatorname{Ric}(\gamma, \gamma), 0\} \colon \gamma \in T_pM,\ |\gamma| = 1,\ p \in \Omega\}.$$

Idea of the proof

Write the Riemannian metric of M (locally) as

$$ds^2 = \sigma_{ij} dx^i dx^j \quad \text{(Einstein summation).}$$

Proof of Rosenberg-Schulze-Spruck estimate

Corresponding local coordinate frame on $\Sigma = \Sigma_u$ is then given by

$$X_i = \partial_i + u_i\partial_t, \quad u_i = \frac{\partial u}{\partial x_i},\ u(x,t) := u(x),\ x \in \Omega.$$

Furthermore, the unit normal field (to Σ) can be written as

$$\eta = \frac{1}{W}(-u^i\partial_i + \partial_t),\ u^i = \sigma^{ij}u_j,\ W := \sqrt{1 + |\nabla u|^2}.$$

So, the induced Riem. metric (from $M \times \mathbb{R}$) on Σ is

$$ds^2_\Sigma = g_{ij}d\tau^i d\tau^j, \quad d\tau^i(X_j) = \delta_{ij},$$
$$g_{ij} = \langle X_i, X_j\rangle = \sigma_{ij} + u_i u_j,$$
$$g^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2}.$$

The minimal graph equation in nondivergence form is then

$$\frac{1}{W}g^{ij}\left(u_{ij} - \Gamma^k_{ij}u_k\right) = \frac{1}{W}g^{ij}D_iD_ju = 0.$$

Proof of Rosenberg-Schulze-Spruck estimate

Recall the "Bochner" formula for $H \equiv 0$.

$$\Delta^{\Sigma}\left(\frac{1}{W}\right) = -\left(\widetilde{\mathrm{Ric}}(\eta) + \|\mathrm{B}\|^2\right)\frac{1}{W}.$$

We have

$$\widetilde{\mathrm{Ric}}(\eta) = \left(1 - W^{-2}\right)\mathrm{Ric}^M(\gamma), \quad \gamma = \frac{\eta^M}{|\eta^M|}, \tag{8}$$

where η^M is the "TM-component" of $\eta \in TM \oplus \mathbb{R}$. [Note: If η is vertical ($\eta^M = 0$), then $W = 1$ and (8) holds trivially.]
Using (8), the standard formula

$$\Delta(g \circ \varphi) = (g' \circ \varphi)\Delta\varphi + (g'' \circ \varphi)|\nabla\varphi|^2,$$

and the "Bochner" formula, we get

$$\Delta^{\Sigma} W = \frac{2|\nabla^{\Sigma} W|^2}{W} + W\|\mathrm{B}\|^2 + W(1 - W^{-2})\,\mathrm{Ric}^M(\gamma).$$

Proof of Rosenberg-Schulze-Spruck estimate

Define (on $\Omega \times \mathbb{R}$) a function $h(x) = \varphi(x)W(x)$, depending only on x-variable of points $(x, t) \in \Omega \times \mathbb{R}$, with

$$\varphi = e^{\alpha u}, \ \alpha \geq 0,$$

and an elliptic second order operator

$$\begin{aligned} Lh &:= \Delta^\Sigma h - 2g^{ij}\frac{X_i W}{W} X_j h = \Delta^\Sigma h - 2g^{ij}\frac{W_i}{W} h_j \\ &= \varphi\left(\Delta^\Sigma W - \frac{2}{W}|\nabla^\Sigma W|^2\right) + W\Delta^\Sigma \varphi \\ &= W\left(\Delta^\Sigma \varphi + \varphi\left(\|\mathrm{B}\|^2 + \left(1 - W^{-2}\right)\mathrm{Ric}^M(\eta)\right)\right). \end{aligned}$$

Since

$$\begin{aligned} \Delta^\Sigma \varphi &= \Delta^\Sigma\left(e^{\alpha u}\right) = \alpha e^{\alpha u}\underbrace{\Delta^\Sigma u}_{=0} + \alpha^2 e^{\alpha u}|\nabla^\Sigma u|^2 = \alpha^2 e^{\alpha u}|\nabla^\Sigma u|^2 \\ &= \alpha^2 e^{\alpha u}\left(1 - W^{-2}\right), \end{aligned}$$

Proof of Rosenberg-Schulze-Spruck estimate

we get

$$Lh = h\left(\|\mathrm{B}\|^2 + \left(1 - W^{-2}\right)\left(\alpha^2 + \mathrm{Ric}^M(\gamma)\right)\right).$$

Choosing α as in the claim, i.e.

$$\alpha^2 = \sup\{\max\{-\mathrm{Ric}(\gamma,\gamma), 0\} \colon \gamma \in T_pM,\ |\gamma| = 1,\ p \in \Omega\},$$

we obtain

$$Lh \geq 0, \quad \text{i.e. } h = e^{\alpha u}W \text{ is a subsolution.}$$

By the Hopf maximum principle

$$\sup_{\Omega} e^{\alpha u}W = \sup_{\partial\Omega} e^{\alpha u}W,$$

and the estimate (7) follows.
The proof of the gradient estimate due to Spruck follows similar lines.