Homotopy and vector bundles
Exercise 5 (9.10.2014)

1. a) Prove that the map (lecture notes, p. 29)

$$\pi|\colon B^n \to S^n \setminus \{e_{n+1}\}$$

is a homeomorphism.
b) See the proof of Proposition 1.5 (p. 30). Prove that

$$U_1 \cap U_2 \approx S^{n-1} \times (-1, 1).$$

2. a) Prove that there doesn't exist an embedding $S^1 \to \mathbb{R}$.
b) Prove that there doesn't exist an embedding $S^2 \to \mathbb{R}^2$.

3. Suppose X is the union of an increasing sequence $U_1 \subset U_2 \subset \dots$ of open simply connected subsets. Prove that X is simply connected.

4. Suppose X is a topological space, $x_0 \in X$ and $f, g\colon X \to S^1$ continuous functions for which $f(x_0) = g(x_0)$ and $f \simeq g$. Prove that $f \simeq g$ rel x_0. [Hint: if $h\colon f \simeq g$, find the new homotopy in the form $H_t = u_t \circ h_t$, where u_t is a suitable rotation of the circle.]

5. What is false in the following proof?
Claim. $\pi(S^2, e_3) = 0$.
"Proof". Let $\alpha \in \Omega(S^2, e_3)$ be a loop. Choose $x \in S^2$ such that $x \notin \alpha(I)$. Now $S^2 \setminus \{x\} \approx \mathbb{R}^2$, which is contractible, so the path α is null homotopic in the space $S^2 \setminus \{x\}$. Thus it is null homotopic also in the space S^2.
6. a) Prove that the system of equations

$$\begin{cases} x\cos y = & x^2 + y^2 - 1 \\ y\cos x = & \sin(2\pi(x^3 + y^3)) \end{cases}$$

has a solution in the disc $\bar{B}^2$. [If you need a hint, you can ask me.]
b) Same question for the system

$$\begin{cases} x\cos y = & x^2 + y^2 - 1 \\ y\cos x = & \tan(2\pi(x^3 + y^3)). \end{cases}$$