Homotopy and vector bundles
Exercise 4 (2.10.2014)

1. Give examples of spaces (X, x_0), (Y, y_0) and functions $f: (X, x_0) \to (Y, y_0)$ such that

a) f is injective, f_* is not injective
b) f is not injective, f_* is injective
c) f is surjective, f_* is not surjective
d) f is not surjective, f_* is surjective.

2. Give an example of a covering map $p: (X, x_0) \to (Y, y_0)$ and paths $\alpha, \beta \in \Omega(Y, y_0)$ such that $\tilde{\alpha}(1) = \tilde{\beta}(1)$, but $\alpha \not\simeq \beta$.

3. Let X be a topological space and $A \subset X$. We say that A is a deformation retract of X, if there exists a homotopy $H: X \times I \to X$ such that

$$H(x, 0) = x \text{ jokaisella } x \in X,$$

$$H(x, 1) \in A \text{ jokaisella } x \in X$$

and

$$H(a, 1) = a \text{ jokaisella } a \in A.$$

We say that A is a strong deformation retract of X, if there exists a homotopy $H: X \times I \to X$ such that

$$H(x, 0) = x \text{ jokaisella } x \in X,$$

$$H(x, 1) \in A \text{ jokaisella } x \in X$$

and

$$H(a, t) = a \text{ jokaisella } a \in A, t \in I.$$

a) Prove: if A is a deformation retract of X, then A is a retract of X.
b) Prove: if A is a deformation retract of X, then $A \simeq X$.
c) Prove that S^{n-1} is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$.
d) Give an example of a situation where A is a deformation retract, but not a strong deformation retract of X.

4. Let X be a topological space, $f, g\colon X \to S^n$ continuous maps and $f(x) + g(x) \neq \bar{0}$ for every $x \in X$. Prove that $f \simeq g$.

5. Suppose that $p\colon X \to Y$ is a covering map and Y is path connected. Prove that every fiber $p^{-1}\{y\}$ has the same cardinality. [Hint: use path lifting]

6. Let $f\colon \bar{B}^2 \to \bar{B}^2$ be a homeomorphism. Prove that $fB^2 = B^2$ and $fS^1 = S^1$. Deduce from this that $\bar{B}^2$ is not a homogeneous space. [Hint: Remove one point]
(A space X is homogeneous, if for every pair x, y of points of X there exists a homeomorphism $X \to X$ mapping x to y.)