Homotopy and vector bundles
Exercise 3 (25.9.2014)

1. Let $Y = \prod_{j \in J} Y_j$ be a product space. Prove that continuous maps $f, g\colon X \to Y$ are homotopic if and only if $pr_j \circ f \simeq pr_j \circ g$ for every $j \in J$. Here $pr_j\colon Y \to Y_j$ is the projection map.

2. Let $n \in \mathbb{N}$, X is the half-ball $\{x \in \bar{B}^n : x_n \geq 0\}$ and $A = \{x \in \bar{B}^n : x_n = 0\}$. Prove that the inclusion $j\colon A \to X$ is a homotopy equivalence. Hint: Use the projection $\mathbb{R}^n \to \mathbb{R}^{n-1} \times \{0\}$.

3. A path $\alpha\colon I \to \mathbb{R}^n$ is called a *polyline* if there exist numbers $0 = t_0 < t_1 < \cdots < t_N = 1$ such that the restriction $\alpha|[t_{j-1}, t_j]$ is affine for all $1 \leq j \leq N$. (Affine means that α is of the form $\alpha(t) = a_j t + b_j$ in each subinterval). Let U be an open subset of $\mathbb{R}^n$ and α a path in U. Prove that α is path homotopic (in U) with some polyline.

4. If σ is a path between points $x_0, x_1 \in X$, then σ induces an isomorphism

$$\sigma_\sharp\colon \pi(X, x_0) \to \pi(X, x_1), \quad \sigma_\sharp(\bar{\alpha}) = \overline{\sigma^{\leftarrow}}\bar{\alpha}\bar{\sigma}.$$

Suppose that $f, g\colon X \to Y$ are homotopic, but not necessarily rel x_0. If we denote $y_0 = f(x_0)$ and $y_1 = g(x_0)$ we have the induced homomorphisms

$$f_*\colon \pi(X, x_0) \to \pi(Y, y_0), \; g_*\colon \pi(X, x_0) \to \pi(Y, y_1).$$

Prove the following connection between f_* and g_*: Let $h\colon f \simeq g$. The formula $\sigma(t) = h(x_0, t)$ defines a path in Y from y_0 to y_1. Then

$$g_* = \sigma_\sharp \circ f_*.$$

[Hint: Let $\alpha \in \Omega(X, x_0)$. We should prove that $g_*(\bar{\alpha}) = \overline{\sigma^{\leftarrow}} f_*(\bar{\alpha})\bar{\sigma}$, that is, $\overline{g \circ \alpha^{\leftarrow}\sigma^{\leftarrow} f \circ \alpha}\bar{\sigma} = \bar{\epsilon}$. Let $z_0 = (1, 1)$ and $\omega \in \Omega(I^2, z_0)$ the composite of four line paths, which goes around the boundary of I^2 counter-clockwise. Denote $F\colon I^2 \to Y$, $F(s, t) = h(\alpha(s), t)$. Deduce that $\omega \sim \epsilon$ in I^2 which implies that $F \circ \omega \sim \epsilon$ in Y which implies the claim.]

5. With help of the previous exercise, prove the following: If $f\colon X \to Y$ is a homotopy equivalence then $f_*\colon \pi(X, x_0) \to \pi(Y, f(x_0))$ is an isomorphism.

6. Prove that a covering map is always a surjective open immersion.