Geometric measure theory, Exercise 8, 20.11.2014, the last exercise of the course

1. Show that for Dirac measures $\delta_{a_j}, a_j \in \mathbb{R}^n$:
$\delta_{a_j} \to \delta_a$ weakly if $a_j \to a \in \mathbb{R}^n$,
$\delta_{a_j} \to 0$ weakly if $|a_j| \to \infty$.

2. Show that $\frac{1}{k}\sum_{j=1}^{k} \delta_{j/k} \to \mathcal{L}^1 \llcorner [0,1]$ weakly when $k \to \infty$.

3. Show that if $\mu_j \to \mu$ weakly and $A \subset \mathbb{R}^n$ is bounded with $\mu(\partial A) = 0$, then $\mu_j(A) \to \mu(A)$.

4. Give an example of a finite Borel measure on $\mathbb{R}$ which has a tangent measure ν for which $0 \notin \operatorname{spt} \nu$.

5. Let $0 < s < \infty$ and suppose that there exists a Borel measure ν on $\mathbb{R}^n$ such that for some positive constants c and d,

$$cr^s \leq \nu(B(x,r)) \leq dr^s \text{ for all } x \in \mathbb{R}^n, 0 < r < 1.$$

Prove that $s = n$.

6. Let μ be a locally finite Borel measure on $\mathbb{R}^n$ such that

$$\operatorname{spt} \mu \subset \{x = (x_1, \dots, x_n) : |x_n| \leq |x|^2\}.$$

Prove that

$$\operatorname{spt} \nu \subset \{x = (x_1, \dots, x_n) : x_n = 0\}$$

for every $\nu \in Tan(\mu, 0)$.