**GEOMETRIC MEASURE THEORY**

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These notes give a sketch of the lectures; definitions, theorems and perhaps some ideas but not many detailed proofs.

## 1. HAUSDORFF MEASURES

The $s$-dimensional *Hausdorff measure* $\mathcal{H}^s$, $s \geq 0$, is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

where, for $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_j \alpha(s) 2^{-s} d(E_j)^s : A \subset \bigcup_j E_j, d(E_j) < \delta\}.$$

Here $\alpha(s)$ is a positive number. For integers $n$, $\alpha(n)$ is the volume of the $n$-dimensional unit ball (with $\alpha(0) = 1$). Then in $\mathbb{R}^n, \mathcal{H}^n = \mathcal{L}^n$, the Lebesgue measure, we shall prove this later.

The *Hausdorff dimension* of $A \subset \mathbb{R}^n$ is

$$\dim A = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

Since (as an easy exercise), $\mathcal{H}^s(A) = 0$ if and only if $\mathcal{H}^s_\infty(A) = 0$, we can replace $\mathcal{H}^s$ in the definition of dim by the simpler $\mathcal{H}^s_\infty$. So, more simply,

$$\dim A = \inf\{s : \forall \varepsilon > 0\ \exists E_1, E_2, \dots \subset X \text{ such that } A \subset \bigcup_j E_j \text{ and } \sum_i d(E_j)^s < \varepsilon\}.$$

For the definition of dimension, the sets $E_j$ above can be restricted to be balls, because each $E_j$ is contained in a ball $B_j$ with $d(B_j) \leq 2d(E_j)$. The spherical measure obtained using balls is not the same as the Hausdorff measure but it is between $\mathcal{H}^s$ and $2^s\mathcal{H}^s$.

The $m$-dimensional Hausdorff measure restricted to a sufficiently nice, even just Lipschitz, surface is the standard surface measure by the area formula which we shall prove later.

The $1/3$ Cantor set has Hausdorff dimension $s = \log 2/\log 3$ with $0 < \mathcal{H}^s(C) < \infty$.

$\mathcal{H}^s$ is a Borel regular outer measure: Borel sets are $\mathcal{H}^s$ measurable and for every $A \subset \mathbb{R}^n$ there is a Borel set $B$ such that $A \subset B$ and $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.

**Theorem 1.1.** [Approximation theorem for Hausdorff measures] Let $A \subset \mathbb{R}^n$ be $\mathcal{H}^s$ measurable with $\mathcal{H}^s(A) < \infty$. Then for every $\varepsilon > 0$ there is a compact set $C$ such that $C \subset A$ and $\mathcal{H}^s(A \setminus C) < \varepsilon$.

This is a special case of the corresponding theorem for general Borel measures. We mean by a *measure* on a set $X$ what is usually meant by outer measure, that is, a non-negative, monotone, countably subadditive function on $\{A : A \subset X\}$ that gives the value $0$ for the empty set. By a *Borel measure* in a metric space $X$ we mean a measure $\mu$ for which Borel sets are measurable and which is *Borel regular* in the sense that for any $A \subset X$ there is a Borel set $B$ such that $A \subset B$ and $\mu(A) = \mu(B)$. A Borel measure $\mu$ is *locally finite* if all compact sets have finite $\mu$ measure.

**Theorem 1.2.** [Approximation theorem for Borel measures] Let $\mu$ be a Borel measure on $\mathbb{R}^n$ with $\mu(\mathbb{R}^n) < \infty$ and let $A \subset \mathbb{R}^n$ be $\mu$ measurable. Then for every $\varepsilon > 0$ there are a compact set $C$ and open set $G$ such that $C \subset A \subset G$ and $\mu(G \setminus C) < \varepsilon$.

Theorem 1.1 follows from Theorem 1.2 by considering the *restriction* measure $\mathcal{H}^s \llcorner A$:

$$\mathcal{H}^s \llcorner A(B) = \mathcal{H}^s(A \cap B) \text{ for } B \subset \mathbb{R}^n.$$

It is a Borel measure when $A$ is $\mathcal{H}^s$ measurable. The proofs of these facts can be found in [Ma], Chapter 1.

Lebesgue density theorem is not valid for Hausdorff measures but we have the following density estimates.

**Definition 1.3.** The $s$-dimensional *upper density* of $A \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is

$$\Theta^{*,s}(A,x) = \limsup_{r \to 0} \frac{\mathcal{H}^s(A \cap B(x,r))}{\alpha(s) r^s},$$

the $s$-dimensional *lower density* of $A \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is

$$\Theta^{s}_{*}(A,x) = \liminf_{r \to 0} \frac{\mathcal{H}^s(A \cap B(x,r))}{\alpha(s) r^s},$$

and the $s$-dimensional *density* of $A \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is

$$\Theta^{s}(A,x) = \lim_{r \to 0} \frac{\mathcal{H}^s(A \cap B(x,r))}{\alpha(s) r^s},$$

if the limit exists.

**Theorem 1.4.** Let $A \subset \mathbb{R}^n$ be $\mathcal{H}^s$ measurable with $\mathcal{H}^s(A) < \infty$. Then

$$2^{-s} \leq \Theta^{*,s}(A,x) \leq 1 \text{ for } \mathcal{H}^s \text{ almost all } x \in A,$$

and

$$\Theta^{*,s}(A,x) = 0 \text{ for } \mathcal{H}^s \text{ almost all } x \in \mathbb{R}^n \setminus A.$$

There are rather easy examples of compact sets $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$ such that $\Theta^s_*(A,x) = 0$ for all $x \in \mathbb{R}^n$.

The proof of Theorem 1.3 can be found in [Ma], Chapter 6. Part of it uses covering theorems, which are very important in GMT. Here are the basic ones, their proofs are also in [Ma], Chapter 2. We denote by $tB$ the ball $B(x,tr)$ when $B = B(x,r)$ and $t > 0$.

**Theorem 1.5.** [$5r$ covering theorem] Let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$ with

$$\sup\{d(B) : B \in \mathcal{B}\} < \infty.$$

Then there are disjoint balls $B_i \in \mathcal{B}$ (countably or finitely many) such that

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_i 5B_i.$$

**Theorem 1.6.** [Vitali's covering theorem] Let $A \subset \mathbb{R}^n$ and let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$ with

$$\inf\{d(B) : x \in B \in \mathcal{B}\} = 0 \text{ for all } x \in A.$$

Then there are disjoint balls $B_i \in \mathcal{B}$ such that

$$\mathcal{L}^n(A \setminus \bigcup_i B_i) = 0.$$

Moreover, if $\varepsilon > 0$, the balls $B_i$ can be chosen so that

$$\sum_i \mathcal{L}^n(B_i) \leq \mathcal{L}^n(A) + \varepsilon.$$

We denote by $\chi_A$ the characteristic function of a set $A$.

**Theorem 1.7.** [Besicovitch's covering theorem] There are positive integers $P(n)$ and $Q(n)$ depending only on $n$ with the following properties. Let $A \subset \mathbb{R}^n$ be a bounded set and let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$ such that every point of $A$ is the centre of some ball in $\mathcal{B}$.

(1) There are balls $B_i \in \mathcal{B}$ such that they cover $A$ and every point of $\mathbb{R}^n$ belongs at most to $P(n)$ balls $B_i$, that is,

$$\chi_A \leq \sum_i \chi_{B_i} \leq P(n).$$

(2) There are families $\mathcal{B}_i \subset \mathcal{B}, i = 1, \dots, Q(n)$, covering $A$ such that each $\mathcal{B}_i$ is disjoint, that is,

$$A \subset \bigcup_{i=1}^{Q(n)} \bigcup_{B\in\mathcal{B}_i} B,$$

and for every $i = 1, \dots, Q(n)$,

$$B \cap B' = \varnothing \text{ for } B, B' \in \mathcal{B}_i, B \neq B'.$$

**Theorem 1.8.** [Vitali's covering theorem for general measures] Let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n$, let $A \subset \mathbb{R}^n$ and let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$ with

$$\inf\{r : x \in B(x, r) \in \mathcal{B}\} = 0 \text{ for all } x \in A.$$

Then there are disjoint balls $B_i \in \mathcal{B}$ such that

$$\mu(A \setminus \bigcup_i B_i) = 0.$$

Moreover, if $\varepsilon > 0$, the balls $B_i$ can be chosen so that

$$\sum_i \mu(B_i) \leq \mu(A) \leq \varepsilon.$$

Observe the difference in Theorems 1.6 and 1.8: for Lebesgue measure we only need that every point of $A$ belongs to arbitrarily small balls of $\mathcal{B}$, but for general measures we need that every point of $A$ is the centre of arbitrarily small balls of $\mathcal{B}$.

Lipschitz maps are very important in GMT: we say that $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, is *Lipschitz* if there is $L < \infty$ such that

$$|f(x) - f(y)| \leq L|x - y| \text{ for all } x, y \in \mathbb{R}^n.$$

The smallest such number $L$ is called the *Lipschitz constant* of $f$ and denoted by $\mathrm{Lip}(f)$.

**Theorem 1.9.** If $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, is Lipschitz and $s \geq 0$, then

$$\mathcal{H}^s(f(A)) \leq Lip(f)^s \mathcal{H}^s(A).$$

Proof exercise.

There is no Fubini theorem for Hausdorff measures, but we have the inequality:

**Theorem 1.10.** If $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, is Lipschitz and $m \leq s \leq n$, then

$$\int^* \mathcal{H}^{s-m}(A \cap f^{-1}\{y\})\, d\mathcal{L}^m y \leq C(m, n, s) Lip(f)^m \mathcal{H}^s(A).$$

Proof, [Ma], Theorem 7.7.

Here $\int^*$ is the upper integral since the integrand need not be measurable. It is measurable if $A$ is closed, or even a Borel set, but that is more difficult to prove.

## 2. Steiner symmetrization and the isodiametric inequality

The reference for this chapter is [EG], pp. 67-78.

Let $V$ be a hyperplane in $\mathbb{R}^n$ through $0$, that is, a linear $(n-1)$-dimensional subspace. We can write any $x \in \mathbb{R}^n$ as $x = x_V + x_V^{\perp}, x \in V, x_V^{\perp} \in V^{\perp}$. Let $A \subset \mathbb{R}^n$. We say that $A$ is symmetric with respect to $V$ if $x_V + x_V^{\perp} \in A$ implies $x_V - x_V^{\perp} \in A$. We say that $A$ is symmetric with respect to the origin if $x \in A$ implies $-x \in A$.

Steiner symmetrization symmetrizes any Lebesgue measurable $A \subset \mathbb{R}^n$ to a Lebesgue measurable set symmetric with respect to a hyperplane without changing the measure and without increasing the diameter.

**Definition 2.1.** Let $a, b \in \mathbb{R}^n$ with $|a| = 1$. Set

$$L_b^a = \{b + ta : t \in \mathbb{R}\},$$

$$P_a = \{x \in \mathbb{R}^n : a \cdot x = 0\}.$$

So $L_b^a$ is the line through $b$ in the direction $a$ and $P_a$ is the hyperplane through the origin orthogonal to $a$.

The *Steiner symmetrization* of a set $A \subset \mathbb{R}^n$ with respect to $P_a$ is

$$S_a(A) = \bigcup_{b \in P_a, A \cap L_b^a \neq \varnothing} \left\{ b + ta : |t| \leq \mathcal{H}^1(A \cap L_b^a)/2 \right\}.$$

**Theorem 2.2.** Let $a \in \mathbb{R}^n$ with $|a| = 1$. For all $A \subset \mathbb{R}^n$,
(1) $d(S_a(A)) \leq d(A)$.
(2) If $A$ is Lebesgue measurable, so is $S_a(A)$ and $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$.

(1) is a rather simple fact based on the definition of $S_a(A)$ and (2) follows from Fubini's theorem.

**Theorem 2.3.** [Isodiametric inequality] For all $A \subset \mathbb{R}^n$,

$$\mathcal{L}^n(A) \leq \alpha(n) 2^{-n} d(A)^n.$$

Notice that there is equality when $A$ is a ball. So ball has the largest measure among sets with a given diameter.

To prove the isodiametric inequality symmetrize $A$ first with respect to the plane orthogonal to the $x_1$-axis, then symmetrize this set with respect to the plane orthogonal to the $x_2$-axis, and so on. After you have symmetrized with respect to all coordinate planes you have a set symmetric with respect to the origin, with the same measure as $A$ and not bigger diameter. So you only need to verify the claim for such symmetric sets, which is rather easy.

Notice that you cannot prove the isodiametric inequality by putting $A$ inside a ball with the same diameter, because for example for equilateral triangle you cannot do that.

The isodiametric inequality leads to the fact that in $\mathbb{R}^n$ Lebesgue measure and $n$-dimensional Hausdorff measure agree:

**Theorem 2.4.** For all $A \subset \mathbb{R}^n$,

$$\mathcal{H}^n(A) = \mathcal{L}^n(A).$$

## 3. BRUNN-MINKOWSKI AND ISOPERIMETRIC INEQUALITY

This chapter is based on [Fe], pp. 273-278.

The sum set of subsets $A$ and $B$ of $\mathbb{R}^n$ is

$$A + B = \{a + b : a \in A, b \in B\}.$$

**Theorem 3.1.** [Brunn-Minkowski inequality] For all $A, B \subset \mathbb{R}^n$,

$$\mathcal{L}^n(A+B)^{1/n} \geq \mathcal{L}^n(A)^{1/n} + \mathcal{L}^n(B)^{1/n}.$$

This is sharp, equality holds when, for example, $A = B = B(0,1)$, then $A + B = B(0,2)$. The inequality is strict quite often. For instance, if $A = B = C$ is the 1/3-Cantor set, the right hand is zero but the left hand side is positive; $C + C = [0,2]$ (exercise).

By approximation the proof is reduced to the case where both $A$ and $B$ are a finite union of intervals in $\mathbb{R}^n$. The proof proceeds then by induction on the total number of these intervals. To use induction hypothesis one splits each of the unions into two collections of intervals by separating with suitable hyperplanes.

The *isoperimetric inequality* says that

$$\mathcal{L}^n(A)^{(n-1)/n} \leq (n\alpha(n)^{1/n})^{-1}P(A).$$

Here $P(A)$ is the perimeter of $A$. The constant here is such that equality occurs for balls.

This inequality has different meanings depending on what perimeter means. The most classical case is where $A$ is a bounded open set with smooth boundary $\partial A$. Then $P(A) = \mathcal{H}^{n-1}(\partial A)$ is the $(n-1)$-dimensional surface area of $\partial A$. A very general case is where $A$ is just a Lebesgue measurable set with finite measure and perimeter is defined in distributional sense. This approach is presented in [EG], Chapter 5.

We shall now interpret perimeter as Minkowski content. This includes the classical case and much more.

**Definition 3.2.** The $(n-1)$-dimensional *Minkowski content* of $A \subset \mathbb{R}^n$ is

$$\mathcal{M}_*^{n-1}(A) = \liminf_{\varepsilon \to 0} \frac{1}{2\varepsilon}\mathcal{L}^n(\{x : dist(x, A) < \varepsilon\}).$$

**Theorem 3.3.** [Isoperimetric inequality] For all $A \subset \mathbb{R}^n$ with $\mathcal{L}^n(\overline{A}) < \infty$,

$$\mathcal{L}^n(A)^{(n-1)/n} \leq (n\alpha(n)^{1/n})^{-1}\mathcal{M}_*^{n-1}(\partial A).$$

Here $\overline{A}$ is the closure of $A$. The proof is based on the Brunn-Minkowski inequality. Observe that $\{x : dist(x, A) < \varepsilon\} = \overline{A} + U(0, \varepsilon)$, where $U(0, \varepsilon)$ is the open ball with centre $0$ and radius $\varepsilon$. One uses Brunn-Minkowski both for this and $\{x : 0 < dist(x, \mathbb{R}^n \setminus A) < \varepsilon\}$.

## 4. LIPSCHITZ MAPS

Recall that $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, is Lipschitz if there is $L < \infty$ such that

$$|f(x) - f(y)| \leq L|x - y| \text{ for all } x, y \in \mathbb{R}^n.$$

We already discussed earlier relations to Haudorff measures. Here we discuss extension and differentiability.

**Theorem 4.1.** [Lipschitz extension] Let $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, be Lipschitz. Then there is a Lipschitz map $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g(x) = f(x)$ for $x \in A$ and $Lip(g) = Lip(f)$.

For $m = 1$ this is easy: $g$ can be defined by the formula

$$g(x) = \inf\{f(y) + Lip(f)|x - y| : y \in A\}, x \in \mathbb{R}^n.$$

When $m > 1$ we can apply this to the coordinate functions of $f$ to get Lipschitz extension $g$ of $f$ with $Lip(g) \leq \sqrt{m}Lip(f)$. For the proof of the full theorem using Zorn's lemma, see [Fe], 2.10.43. This is called Kirszbraun's theorem.

Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ are of bounded variation and hence can be written as a difference of two increasing functions. Thus by Lebesgue's classical theorem they are differentiable almost everywhere. The proof given in [EG], [Fe] and [Ma] of the following theorem takes this as the starting point:

**Theorem 4.2.** [Rademacher's theorem] Let $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, be Lipschitz. Then $f$ is differentiable almost everywhere.

Considering coordinate functions the proof is reduced to the case $m = 1$. By the case $n = 1$ and Fubini's theorem the partial derivates and the gradient $\nabla f = (\partial_1 f, \ldots, \partial_n f)$ exist almost everywhere. Moreover by the case $n = 1$ for every unit vector $e \in S^{n-1}$,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - e \cdot \nabla f(x) = 0$$

for almost all $x \in \mathbb{R}^n$. Hence this holds almost everywhere for a countable dense set of $e \in S^{n-1}$. Differentiability at $x$ means that this holds for every unit vector $e$ and uniformly in $e$. Lipschitz condition is used to verify this.

## 5. Area and coarea formulas

This chapter is mainly based on [EG], Chapter 3, but these can be found also in [Fe], [LY] and [Si].

If $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then

$$\mathcal{L}^n(L(A)) = |\det L| \mathcal{L}^n(A).$$

For diagonal linear maps this is clear. $L$ is diagonal if there exist $\lambda_1, \ldots, \lambda_n$ such that

$$L(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n).$$

Then $\det L = \lambda_1 \cdots \lambda_n$. For general linear maps $L$ it follows by diagonalization: by linear algebra we can write $L$ as

$$L = O_1 \circ D \circ O_2,$$

where $D$ is diagonal and $O_1$ and $O_2$ are orthogonal: $O_j x \cdot O_j y = x \cdot y$ for all $x, y \in \mathbb{R}^n$. The orthogonal maps have determinant $\pm 1$, so $|\det L| = |\det D|$, and orthogonal maps don't change distances, whence they preserve Lebesgue measure. This leads to the basic change of variable formula for Lebesgue measure: if $f : U \to \mathbb{R}^n, U \subset \mathbb{R}^n$ open, is continuously differentiable and injective, then

$$\int_A (g \circ f) J_f = \int_{f(A)} g$$

when $A \subset U$ is Lebesgue measurable and, for example, $g$ is a non-negative Borel function. Here $J_f$ is the Jacobian of $f$:

$$J_f(x) = |\det Df(x)|.$$

We formulate this and later results for non-negative Borel functions, since then the integrals exist, but they may be infinite. For general functions the formulas hold provided the integrals of the positive and negative parts are finite.

Our goal is to prove such formulas for maps $f : \mathbb{R}^n \to \mathbb{R}^m$ between spaces of different dimensions. We shall also prove these for Lipschitz maps. Recall that

Lipschitz maps can always be extended, so it suffices to consider those defined on the whole space.

The area formula deals with the case $f : \mathbb{R}^n \to \mathbb{R}^m, n \leq m,$ including also $m = n$. The starting point again is what happens with linear maps.

**Proposition 5.1.** Let $L : \mathbb{R}^n \to \mathbb{R}^m, n \leq m,$ be linear. Then there are a diagonal map $D : \mathbb{R}^n \to \mathbb{R}^n$ and orthogonal maps $O_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $O_2 : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$L = O_1 \circ D \circ O_2.$$

This follows by basic linear algebra.

**Definition 5.2.** Let $L : \mathbb{R}^n \to \mathbb{R}^m, n \leq m,$ be linear. The Jacobian of $L$ is

$$J_L = |\det D|,$$

where $D$ is as in the above decomposition.

It follows from the above that for any $A \subset \mathbb{R}^n$,

$$\mathcal{H}^n(L(A)) = J_L \mathcal{L}^n(A).$$

The following formula shows that $J_L$ is independent of the decomposition used. We denote by $L^* : \mathbb{R}^m \to \mathbb{R}^n$ the adjoint of $L$ defined by $L(x) \cdot y = x \cdot L^*(y)$ for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

**Proposition 5.3.** Let $L : \mathbb{R}^n \to \mathbb{R}^m, n \leq m,$ be linear. Then

$$J_L^2 = \det(L^* \circ L).$$

This also is rather easy linear algebra.

For any map $f : \mathbb{R}^n \to \mathbb{R}^m$ and $A \subset \mathbb{R}^n$ define the multiplicity function $N(f, A, \cdot)$ by

$$N(f, A, y) = \#\{x \in A : f(x) = y\} = \mathcal{H}^0(A \cap f^{-1}\{y\}).$$

If $f$ is differentiable at $x$, we define the Jacobian of $f$ at $x$ by

$$J_f(x) = J_{Df(x))}.$$

Here the matrix elements of $Df(x)$ are the partial derivatives $\partial_i f_j(x)$.

**Theorem 5.4.** [Area formula] Let $f : \mathbb{R}^n \to \mathbb{R}^m, n \leq m,$ be Lipschitz and $A \subset \mathbb{R}^n$ Lebesgue measurable. Then

$$\int_A J_f \, d\mathcal{L}^n = \int_{\mathbb{R}^m} N(f, A, y) d\,\mathcal{H}^n y.$$

In particular, if $f$ is injective,

$$\mathcal{H}^n(f(A)) = \int_A J_f \, d\mathcal{L}^n.$$

For linear maps this follows from Definition 5.2. If $f$ is continuously differentiable and injective one can prove this decomposing the domain of $f$ into small subdomains where approximation by the differential map is very good and applying the linear case. By Rademacher's theorem the same idea applies to Lipschitz maps but details become much more complicated.

**Theorem 5.5.** [Change of variable formula] Let $f : \mathbb{R}^n \to \mathbb{R}^m, n \leq m,$ be Lipschitz and $g : \mathbb{R}^n \to \mathbb{R}$ a non-negative Borel function. Then

$$\int_{\mathbb{R}^n} g J_f \, d\mathcal{L}^n = \int_{\mathbb{R}^m} \left( \sum_{x : f(x) = y} g(x) \right) d\,\mathcal{H}^n y.$$

Moreover, if $A \subset \mathbb{R}^n$ is Lebesgue measurable and $h : \mathbb{R}^m \to \mathbb{R}$ is a non-negative Borel function,

$$\int_A (h \circ f) J_f \, d\mathcal{L}^n = \int_{f(A)} h(y) N(f, A, y) d\,\mathcal{H}^n y.$$

The coarea formula deals with the case $f : \mathbb{R}^n \to \mathbb{R}^m, m \leq n$. Now we have for linear maps

**Proposition 5.6.** Let $L : \mathbb{R}^n \to \mathbb{R}^m, m \leq n,$ be linear. Then there are a diagonal map $D : \mathbb{R}^m \to \mathbb{R}^m$ and orthogonal maps $O_1 : \mathbb{R}^m \to \mathbb{R}^n$ and $O_2 : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$L = O_2 \circ D \circ O_1^*.$$

This is again basic linear algebra.

**Definition 5.7.** Let $L : \mathbb{R}^n \to \mathbb{R}^m, m \leq n,$ be linear. The Jacobian of $L$ is

$$J_L = |\det D|,$$

where $D$ is as in the above decomposition.

We again have another formula:

**Proposition 5.8.** Let $L : \mathbb{R}^n \to \mathbb{R}^m, m \leq n,$ be linear. Then

$$J_L^2 = \det(L \circ L^*).$$

The coarea formula linear maps $L : \mathbb{R}^n \to \mathbb{R}^m, m \leq n,$ follows easily from the above: if $A \subset \mathbb{R}^n$ is Lebesgue measurable, then

$$J_L \mathcal{L}^n(A) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}(y)) d\,\mathcal{L}^m y.$$

The Jacobian of $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $x$ of differentiability is again defined by

$$J_f(x) = J_{Df(x))}.$$

The most important case is where $f$ is real valued, $m = 1$. Then $J_f(x) = |\nabla f(x)| = \sqrt{\partial_1 f(x)^2 + \cdots + \partial_n f(x)^2}$.

**Theorem 5.9.** [Coarea formula] Let $f : \mathbb{R}^n \to \mathbb{R}^m, m \leq n,$ be Lipschitz and $A \subset \mathbb{R}^n$ Lebesgue measurable. Then

$$\int_A J_f \, d\mathcal{L}^n = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) d\,\mathcal{L}^m y.$$

**Theorem 5.10.** [Change of variable formula] Let $f : \mathbb{R}^n \to \mathbb{R}^m, m \leq n,$ be Lipschitz and $g : \mathbb{R}^n \to \mathbb{R}$ a non-negative Borel function. Then

$$\int_{\mathbb{R}^n} g J_f \, d\mathcal{L}^n = \int_{\mathbb{R}^m} \left( \int_{f^{-1}(y)} g \, d\mathcal{H}^{n-m} \right) d\,\mathcal{L}^m y.$$

## 6. Rectifiable sets

This topic is treated in [Fe], [LY], [Ma] and [Si], and only for one-dimensional sets in [Fa].

**Definition 6.1.** Let $0 < m \leq n$ be integers. A set $E \subset \mathbb{R}^n$ is *m rectifiable* if there are Lipschitz maps $f_j : \mathbb{R}^m \to \mathbb{R}^n, j = 1, 2, \dots,$ such that

$$\mathcal{H}^m(E \setminus \bigcup_j f_j(\mathbb{R}^m)) = 0.$$

Since Lipschitz maps can be extended, the above is equivalent to: $E \subset \mathbb{R}^n$ is $m$ rectifiable if there are Lipschitz maps $f_j : A_j \to \mathbb{R}^n, A_j \subset \mathbb{R}^m, j = 1, 2, \dots,$ such that

$$\mathcal{H}^m(E \setminus \bigcup_j f_j(A_j)) = 0.$$

Moreover, since every subset of $\mathbb{R}^m$ is a countable union of bounded sets, we can require the sets $A_j$ to be bounded. Thus $E \subset \mathbb{R}$ is $m$ rectifiable if and only it can be written as

$$E = \bigcup_j f_j(A_j) \cup E_0,$$

where $A_j \subset \mathbb{R}^m$ are bounded sets, $f_j : A_j \to \mathbb{R}^n$ are Lipschitz and $E_0 \subset \mathbb{R}^n$ with $\mathcal{H}^m(E_0) = 0$. In case $E$ is $\mathcal{H}^m$ measurable we can take the sets $A_j$ to be Borel sets.

The terminology differs in different sources. Often one also requires that $\mathcal{H}^m(E) < \infty$ and calls sets as above countably rectifiable.

The following theorem gives some basic simple properties of rectifiable sets:

**Theorem 6.2.** Let $E \subset \mathbb{R}^n$.

(i) If $E_1 \subset E_2$ and $E_2$ is $m$ rectifiable, then $E_1$ is $m$ rectifiable.
(ii) If $\mathcal{H}^m(E) = 0$, then $E$ is $m$ rectifiable.
(iii) If $E \subset \mathbb{R}^m$, then $E$ is $m$ rectifiable.
(iv) If $E_1, E_2, \dots$ are $m$ rectifiable, then $\cup_j E_j$ is $m$ rectifiable.
(v) If for every $\varepsilon > 0$ there is an $m$ rectifiable set $F \subset E$ such that $\mathcal{H}^m(E \setminus F) < \varepsilon$, then $E$ is $m$ rectifiable.
(vi) If $E$ is $m$ rectifiable, then there is an $m$ rectifiable Borel set $B$ such that $E \subset B$ and $\mathcal{H}^m(B) = \mathcal{H}^m(E)$.

(vii) If $E$ is $m$ rectifiable, it has $\sigma$-finite $\mathcal{H}^m$ measure:
$E = \cup_{j=1}^{\infty} E_j$ with $\mathcal{H}^m(E_j) < \infty$.

**Definition 6.3.** Let $0 < m \leq n$ be integers. A set $F \subset \mathbb{R}^n$ is *purely m unrectifiable* if

$$\mathcal{H}^m(F \cap E) = 0$$

for every $m$ rectifiable set $E \subset \mathbb{R}^n$.

**Theorem 6.4.** Let $A \subset \mathbb{R}^n$ with $\mathcal{H}^m(A) < \infty$. Then there is an $m$ rectifiable Borel set $B$ such that $A \setminus B$ is purely $m$ unrectifiable. Thus $A$ has the decomposition into an $m$ rectifiable set $E$ and a purely $m$ unrectifiable set $F$:

$$A = E \cup F, E \cap F = \varnothing.$$

The proof is easy: let $M$ be the supremum of $\mathcal{H}^m(A \cap B)$ when $B$ ranges over all $m$ rectifiable Borel subsets of $\mathbb{R}^n$, choose $B_j$ with $\mathcal{H}^m(A \cap B_j) > M - 1/j$ and $B = \cup_j B_j$.

Of course the above decomposition is not unique, since we can freely move sets of $\mathcal{H}^m$ measure zero between $E$ and $F$, but it is unique up to sets of $\mathcal{H}^m$ measure zero.

Examples:

*Example* 6.5. Any rectifiable curve $\Gamma = \alpha([a, b])$ (here $\alpha : [a, b] \to \mathbb{R}^n$ is a rectifiable path as in Exercises 4) is 1 rectifiable. $m$ dimensional surfaces with $C^1$ (or even local Lipschitz) parametrizations are $m$ rectifiable.

For one-dimensional sets the rectifiability can be defined in terms of rectifiable curves: $E \subset \mathbb{R}^n$ is 1 rectifiable if and only if there are rectifiable curves $\Gamma_j, j = 1, 2, \ldots,$ such that

$$\mathcal{H}^1(E \setminus \bigcup_j \Gamma_j) = 0.$$

*Example* 6.6. Let $E = \bigcup_{j=1}^{\infty} S_j$ where $S_j$ is the circle $S_j = \{x \in \mathbb{R}^2 : |x - q_j| = 2^{-j}\}$ and the $q_j$ are all the points in $\mathbb{R}^2$ with rational coordinates. Then $E$ is 1 rectifiable with $\mathcal{H}^1(E) < \infty$. Anyway, $E$ is dense in $\mathbb{R}^2$.

*Example* 6.7. Let $C(1/4) = C \times C$ where $C \subset [0, 1]$ is the standard Cantor set where one deletes each time in the construction half of the previous interval. Then $0 < \mathcal{H}^{1/2}(C) < \infty$, $0 < \mathcal{H}^1(C(1/4)) < \infty$ and $C(1/4)$ is purely 1 rectifiable.

Rectifiability can be defined in terms of $C^1$ maps instead of Lipschitz maps. This is based on the following theorem:

**Theorem 6.8.** Let $f : \mathbb{R}^m \to \mathbb{R}$ be Lipschitz. If $\varepsilon > 0$, then there is a $C^1$ mapping $g : \mathbb{R}^m \to \mathbb{R}$ such that

$$\mathcal{L}^m(\{x \in \mathbb{R}^m : g(x) \neq f(x) \text{ or } \nabla g(x) \neq \nabla f(x)\}) < \varepsilon.$$

The proof is based on Rademacher's theorem and Whitney's extension theorem, see [EG], Sections 6.5 and 6.6. From this we get

**Theorem 6.9.** A set $E \subset \mathbb{R}^n$ is $m$ rectifiable if and only if there are $C^1$ maps $f_j : \mathbb{R}^m \to \mathbb{R}^n, j = 1, 2, \dots,$ such that

$$\mathcal{H}^m(E \setminus \bigcup_j f_j(\mathbb{R}^m)) = 0.$$

Using the area formula, and parts of its proof, one obtains injective $C^1$ maps $f_j : \mathbb{R}^m \to \mathbb{R}^n$ such that their and their inverses' Lipschitz constants are arbitrarily close to one. Then one also finds that $E \subset \mathbb{R}^n$ is $m$ rectifiable if and only if there are $C^1$ submanifolds $M_j$ of $\mathbb{R}^n$ such that

$$\mathcal{H}^m(E \setminus \bigcup_j M_j) = 0.$$

Using $C^2$ maps leads to a strictly smaller class of sets.

## 7. Rectifiable sets and approximate tangent planes

Our next goal is to characterize rectifiability in terms of almost everywhere existing tangent planes. Because of Example 6.6 we cannot use ordinary tangent planes but we need approximate tangent planes. First some notation. Let $0 < m < n$ be integers. Set

$$G(n, m) = \{V : V \text{ is an } m-\text{ dimensional linear subspace of } \mathbb{R}^n\}.$$

So $V \in G(n, m)$ if it is an $m$-dimensional plane through the origin. $G(n, m)$ is called Grassmannian manifold. It is an $m(n-m)$-dimensional smooth submanifold (surface) in the $n^2$-dimensional linear space of all linear maps ($n \times n$ matrices) $\mathbb{R}^n \to \mathbb{R}^n$. Later we shall make it into a compact metric space and put a natural measure on it, but so far we don't need them.

For $V \in G(n, m), \eta > 0$ and $a \in \mathbb{R}^n$ we define the cone around $V + a$:

$$X(a, V, \eta) = \{x \in \mathbb{R}^n : d(x - a, V) < \eta|x - a|\}.$$

When $m = 1, V$ is a line and this is just a two-sided sector around $V + a$ (draw a picture, also when $n = 3, m = 2$).

Recall the upper density

$$\Theta^{*,m}(A, x) = \limsup_{r \to 0} \frac{\mathcal{H}^m(A \cap B(x, r))}{\alpha(m) r^m},$$

and its basic inequalities from Theorem 1.4. In particular the following consequence of it will be behind several arguments below: if $A$ and $B$ are disjoint $\mathcal{H}^m$ measurable sets with $\mathcal{H}^m(A) < \infty$ and $\mathcal{H}^m(B) < \infty$, then $\Theta^{*,m}(B, x) = 0$ for $\mathcal{H}^m$ almost $x \in A$.

**Definition 7.1.** Let $A \subset \mathbb{R}^n, V \in G(n, m)$ and $a \in \mathbb{R}^n$.

We say that $V$ is an *ordinary tangent $m$ plane* for $A$ at $a$ if for all $\eta > 0$ we have for all sufficiently small $r > 0$,

$$A \cap B(a, r) \setminus X(a, V, \eta) = \varnothing.$$

We say that $V$ is an *approximate tangent $m$ plane* for $A$ at $a$ if $\Theta^{*,m}(A,a) > 0$ and for all $\eta > 0$,

$$\lim_{r\to 0} r^{-m}\mathcal{H}^m(A \cap B(a,r) \setminus X(a,V,\eta)) = 0.$$

Notice that we defined this so that $V$ is linear, that is, it goes through the origin. The actual geometric tangent plane through $a$ is then $V + a$.

The density statement before Definition 7.1 now gives: if $A$ and $B$ are disjoint $\mathcal{H}^m$ measurable sets with $\mathcal{H}^m(A) < \infty$ and $\mathcal{H}^m(B) < \infty$, then for $\mathcal{H}^m$ almost $a \in A$ an approximate tangent $m$ plane for $A$ at $a$ (if it exists) is also an approximate tangent $m$ plane for $A \cup B$ at $a$.

Our goal now is to prove the following theorem:

**Theorem 7.2.** Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$. Then $E$ is $m$ rectifiable if and only if $E$ has an approximate tangent $m$ plane at $\mathcal{H}^m$ almost all points $a \in E$.

The fact that an $m$ rectifiable set has an approximate tangent $m$ plane almost everywhere is easier, in particular if use the characterization of rectifiablity in terms of $C^1$ surfaces. Then we only need to know that $C^1$ surfaces have ordinary tangent plane at all of their points and use the following proposition:

**Proposition 7.3.** Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$. Suppose that $E = \cup_{j=1}^{\infty} E_j$ where the $E_j$ are $\mathcal{H}^m$ measurable sets which have an approximate tangent $m$ plane at $\mathcal{H}^m$ almost all points $a \in E_j$. Then $E$ has an approximate tangent $m$ plane at $\mathcal{H}^m$ almost all points $a \in E$.

This is a rather easy consequence of the definition of approximate tangent $m$ plane and of the statement before Theorem 7.2.

We use the following notation when $V \in G(n, n-m)$:

$$P_V : \mathbb{R}^n \to V \text{ is the orthogonal projection},$$

$$Q_V : \mathbb{R}^n \to V^{\perp} \text{ is the orthogonal projection}.$$

Notice that we changed the role of $V$: instead of considering $m$ dimensional planes, which could be approximate tangent planes, we now consider $n-m$ dimensional planes, which could be orthogonal complements of approximate tangent planes.

For the converse direction in Theorem 7.2 the key lemma is

**Lemma 7.4.** Suppose $E \subset \mathbb{R}^n, V \in G(n, n-m), 0 < \eta < 1$ and $0 < r \leq \infty$. If

$$E \cap B(a,r) \cap X(a,V,\eta) = \varnothing \text{ for all } a \in E,$$

then $E$ is $m$ rectifiable.
Here $B(a,\infty) = \mathbb{R}^n$.

Think about this when $n = 2, m = 1$, and draw a picture. You should see that when $r = \infty$, the assumption means that $Q_V|E$ is injective with Lipschitz inverse $f = (Q_V|E)^{-1}$. Then $E = f(Q_V(E))$ which is $m$ rectifiable. The proof is reduced to the case $r = \infty$ composing $E$ into a union of sets of diameter less than $r$.

This lemma already is enough to finish the proof for a large class of sets; those for which there is $c > 0$ such that $\mathcal{H}^m(E \cap B(a,r)) > cr^m$ for $a \in E, 0 < r < 1$. The Cantor set $C(1/4)$ in Example 6.7 is one of those. With a little more thought we could also do the sets with positive lower density: $\Theta^m_*(E,a) > 0$ for $a \in E$. To following lemma takes care of the sets for which $\Theta^m_*(E,a) = 0$. It extends Lemma 7.4 from the case where the cones contain no points of $E$ to the case where they contain very little of $E$; you should think of $\lambda$ as a small number.

**Lemma 7.5.** Suppose $E \subset \mathbb{R}^n, V \in G(n, n-m), 0 < \eta < 1, \delta > 0$ and $\lambda > 0$. If $E$ is purely $m$ unrectifiable and

$$\mathcal{H}^m(E \cap B(a,r) \cap X(a,V,\eta)) \leq \lambda r^m \eta^m \text{ for all } a \in E, 0 < r < \delta,$$

then

$$\mathcal{H}^m(E \cap B(a,\delta/6)) \leq c\lambda\delta^m \text{ for all } a \in \mathbb{R}^n,$$

where $c > 0$ depends only on $m$.

Combining this lemma with the density inequality $\Theta^{*,m}(E,a) \geq 2^{-m}$ for $\mathcal{H}^m$ almost all $a \in E$ of Theorem 1.4 we obtain

**Theorem 7.6.** Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$. Then $E$ is purely $m$ unrectifiable if and only if $E$ does not have any approximate tangent $m$ plane at $\mathcal{H}^m$ almost all points $a \in E$.

Theorem 7.2 easily follows from this. We can use Lemma 7.5 to get much sharper information about purely $m$ unrectifiable sets. They do not only fail to have approximate tangent planes, that is, fail to be concentrated near $m$ planes, but they are scattered in all directions:

**Theorem 7.7.** Suppose $E \subset \mathbb{R}^n, V \in G(n, n-m), 0 < \eta < 1$ and $E$ is purely $m$ unrectifiable with $\mathcal{H}^m(E) < \infty$. Then

$$\Theta^{*,m}(E \cap X(a,V,\eta), a) \geq c\eta^m \text{ for } \mathcal{H}^m \text{ almost all } a \in E,$$

where $c > 0$ only depends on $m$.

## 8. Rectifiable sets and orthogonal projections

We again use the following notation when $V \in G(n,m)$:

$$P_V : \mathbb{R}^n \to V \text{ is the orthogonal projection,}$$

$$Q_V : \mathbb{R}^n \to V^\perp \text{ is the orthogonal projection.}$$

Our goal here is to prove the Besicovitch-Federer projection theorem:

**Theorem 8.1.** Let $A \subset \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(A) < \infty$. Then
(1) $A$ is purely $m$ unrectifiable if and only if $\mathcal{H}^m(P_V(A)) = 0$ for almost all $V \in G(n,m)$,
(2) $A$ is $m$ rectifiable if and only if for all $F \subset A$ with $\mathcal{H}^m(F) > 0$ we have $\mathcal{H}^m(P_V(F)) > 0$ for almost all $V \in G(n,m)$.

This is due to Besicovitch for $m = 1, n = 2$ from 1939 and due to Federer for general dimensions from 1947. The proof can be found in the books [Fe], [LY] and [Ma], and for $m = 1, n = 2$ in [Fa].

(1) and (2) are easily seen to be equivalent as a consequence of the decomposition theorem 6.4. For (1) the essential part is that $\mathcal{H}^m(P_V(A)) = 0$ for almost all $V \in G(n,m)$ if $A$ is purely $m$ unrectifiable. The other part follows from "$\mathcal{H}^m(P_V(F)) > 0$ for almost all $V \in G(n,m)$ if $F$ is $m$ rectifiable and $\mathcal{H}^m(F) > 0$". This is reduced to the case where $F$ is a subset of a $C^1$ surface. Then one can in fact show more: for any orthonormal coordinate system in $\mathbb{R}^n$ there is some $m$-dimensional coordinate plane on which $F$ projects onto a set of positive measure. This implies the above statement. In particular in the plane a 1-rectifiable set of positive measure can project into a set of zero length in at most one direction.

What does 'almost all $V \in G(n,m)$' mean? For $m = 1$ it is clear: lines through the origin can be identified with the pair of the points where such line intersects the unit sphere $S^{n-1}$. Thus we can define a measure $\gamma_{n,1}$ on $G(n,1)$ by

$$\gamma_{n,1}(A) = c(n)\mathcal{H}^{n-1}(\bigcup_{L\in A} L \cap S^{n-1}) \text{ for } A \subset G(n,1).$$

For the hyperplanes $V \in G(n, n-1)$ we can consider their orthogonal complements $V^\perp \in G(n,1)$ and define

$$\gamma_{n,n-1}(A) = c(n)\mathcal{H}^{n-1}(\bigcup_{V\in A} V^\perp \cap S^{n-1}) \text{ for } A \subset G(n,n-1).$$

We choose here $c(n) = 1/\mathcal{H}^{n-1}(S^{n-1})$ so that $\gamma_{n,1}$ and $\gamma_{n,n-1}$ are probability measures. Then 'almost all $V \in G(n,m)$' for $m = 1$ and $m = n-1$ mean almost all with respect to $\gamma_{n,1}$ and $\gamma_{n,n-1}$. We could also define this concept for $1 < m < n-1$ in an elementary fashion but it is better to use the theory of Haar measure.

Let $O(n)$ be the orthogonal group of $\mathbb{R}^n$. This means that $g \in O(n)$ if and only if $g$ is a linear map of $\mathbb{R}^n$ onto itself which preserves the inner product:

$$g(x) \cdot g(y) = x \cdot y \text{ for all } x, y \in \mathbb{R}^n.$$

Then $O(n)$ is a group with composition $g \circ h$ as the group operation (or matrix multiplication). Any metric on the space of all linear maps $L : \mathbb{R}^n \to \mathbb{R}^n$ induces a metric on $O(n)$. For example, we can use the operator norm

$$d(L_1, L_2) = \sup\{|L_1x - L_2x| : |x| \le 1\}.$$

This makes $O(n)$ into a compact metric group: the operations $(g,h) \mapsto g \circ h$ and $g \mapsto g^{-1}$ are continuous. The theory of Haar measure tells us that there exists a unique invariant Borel probability measure $\theta_n$ on $O(n)$. This means that $\theta_n(O(n)) = 1$ and

$$\theta_n(gA) = \theta_n(Ag) = \theta_n(A) \text{ for all } g \in O(n), A \subset O(n),$$

where $gA = \{gh : h \in A\}, Ag = \{hg : h \in A\}$. Fixing and arbitrary $V_0 \in G(n,m)$ we can now define

$$\gamma_{n,m}(A) = \theta_n(\{g \in O(n) : g(V_0) \in A\}) \text{ for } A \subset G(n,m).$$

Observe that $O(n)$ acts transitively on $G(n,m)$: for any $V,W \in G(n,m)$ there are (two if $m=1, n=2$, infinitely many otherwise) $g \in O(n)$ such that $g(V)=W$. By the invariance of $\theta_n$ this definition does not depend on $V_0$. It agrees with the previous one when $m=1$ and $m=n-1$. It follows that $\gamma_{n,m}$ is invariant under $O(n)$:

$$\gamma_{n,m}(A) = \gamma_{n,m}(\{g(V) : V \in A\} \text{ for } g \in O(n), A \subset G(n,m).$$

Moreover, it is the only Borel probability measure on $G(n,m)$ with this property. From the definition we see that

$$\gamma_{n,m}(A) = \gamma_{n,n-m}(\{V^\perp : V \in A\}) \text{ for } A \subset G(n,m).$$

Now we know what the statement of Theorem 8.1 means and we start proving the essential part of it: we assume that $A \subset \mathbb{R}^n$ is $\mathcal{H}^m$ measurable and purely $m$ unrectifiable with $\mathcal{H}^m(A) < \infty$ and we want to prove that $\mathcal{H}^m(P_V(A)) = 0$ for $\mathcal{H}^m$ almost all $V \in G(n,m)$. The structure of the proof is the following. We again switch in notation to $(n-m)$-planes $V \in G(n,n-m)$, because the orthogonal complements will be more essential than the planes on which we project. For a given $V \in G(n,n-m)$ we consider three subsets of points $a \in A$ according to how $A$ is distributed near $V+a$. We show that for every $V \in G(n,n-m)$ all these subsets project to zero $m$-dimensional measure. Then we show that for almost all $V$ these three subset cover almost all of $A$. This will complete the proof. The last step is done first in the case $m=n-1$ using the more concrete representation of $\gamma_{n,n-1}$ in terms of the surface measure on $S^{n-1}$. The general case is reduced to this using the definition of $\gamma_{n,m}$ and some Fubini type arguments.

Now we define the three subsets of $A$. Recall the cones

$$X(a,V,\eta) = \{x \in \mathbb{R}^n : d(x-a,V) < \eta|x-a|\} = \{x \in \mathbb{R}^n : |Q_V(x-a)| < \eta|x-a|\}$$

from the previous chapter. Let $\delta > 0$ and $V \in G(n,n-m)$. Set

$$A_{1,\delta}(V) = \{a \in A : \limsup_{\eta\to 0} \sup_{0<r<\delta} (\eta r)^{-m}\mathcal{H}^m(A \cap B(a,r) \cap X(a,V,\eta)) = 0\},$$

$$A_{2,\delta}(V) = \{a \in A : \limsup_{\eta\to 0} \sup_{0<r<\delta} (\eta r)^{-m}\mathcal{H}^m(A \cap B(a,r) \cap X(a,V,\eta)) = \infty\},$$

$$A_3(V) = \{a \in A : \#(A \cap (V+a)) = \infty\}.$$

**Lemma 8.2.** $\mathcal{H}^m(A_{1,\delta}(V)) = 0.$

**Lemma 8.3.** $\mathcal{H}^m(Q_V(A_{2,\delta}(V))) = 0.$

**Lemma 8.4.** $\mathcal{H}^m(Q_V(A_3(V)) = 0.$

The pure unrectifiability is only needed in the first lemma. Its proof follows rather easily from Lemma 7.5. Lemma 8.3 follows with an application of Vitali's covering theorem on $V^\perp$. Lemma 8.4 follows when one applies Theorem 1.10 to the projection $Q_V$ with $s=m$.

The next lemma is the most essential and difficult part of the proof:

**Lemma 8.5.** Let $A \subset \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(A) < \infty$ and $\delta > 0$. Then for $\gamma_{n,n-m}$ almost all $V \in G(n, n-m)$ the following holds: for $\mathcal{H}^m$ almost all $a \in A$ either

$$\limsup_{\eta \to 0} \sup_{0<r<\delta} (\eta r)^{-m} \mathcal{H}^m(A \cap B(a,r) \cap X(a,V,\eta)) = 0,$$

or

$$\limsup_{\eta \to 0} \sup_{0<r<\delta} (\eta r)^{-m} \mathcal{H}^m(A \cap B(a,r) \cap X(a,V,\eta)) = \infty,$$

or

$$(A \setminus \{a\}) \cap (V + a) \cap B(a, \delta) \neq \varnothing.$$

Theorem 8.1 follows combining these four lemmas.

To prove Lemma 8.5 for $m = n-1$ one considers the (outer) measure $\Psi$ on $S^{n-1}$:

$$\Psi(B) = \sup_{0<r<\delta} r^{1-n} \mathcal{H}^{n-1}(A \cap B(0,r) \cap (\cup_{v \in B} L_v)), B \subset S^{n-1},$$

where $L_v = \{uv : u \in \mathbb{R}\}$. This is badly non-additive, Borel sets are not $\Psi$ measurable, but it is countably subadditive. The proof is then reduced to the following: for $\mathcal{H}^{n-1}$ almost all $v \in S^{n-1}$ either

$$\limsup_{\tau \to 0} t^{1-n} \Psi(S^{n-1} \cap B(v,t)) = 0,$$

or

$$\limsup_{\tau \to 0} t^{1-n} \Psi(S^{n-1} \cap B(v,t)) = \infty,$$

or

$$(A \setminus \{0\}) \cap L_v \cap B(0, \delta) \neq \varnothing.$$

To establish this one can prove the following theorem for measures (that is, outer measures) on $\mathbb{R}^k$:

**Theorem 8.6.** Let $\Psi$ be a measure on $\mathbb{R}^k$ and $E$ a Lebesgue measurable set such that $\Psi(E) = 0$. Then for $\mathcal{L}^k$ almost all $x \in E$ either $\limsup_{r \to 0} r^{-k} \Psi(B(x,r)) = 0$ or $\limsup_{r \to 0} r^{-k} \Psi(B(x,r)) = \infty$.

We use this on $R^{n-1}$ which is locally like $S^{n-1}$. If $\Psi$ were a finite *Borel* measure, we would know more: then by the general differentiation theory of measures, or by a quick application of Vitali's covering theorem, for $\mathcal{L}^k$ almost $x \in E, \limsup_{r \to 0} r^{-k} \Psi(B(x,r)) = 0$.

The proof of Theorem 8.6 is based on the Lebesgue density theorem and it is not very difficult.

## 9. Tangent measures and densities

This topic is discussed in [Ma] and [LY].

### Weak convergence

**Definition 9.1.** The sequence $(\mu_j)$ of Borel measures on $\mathbb{R}^n$ *converges weakly* to a Borel measure $\mu$ if for all $\varphi \in C_0(\mathbb{R}^n)$,

$$\int \varphi \, d\mu_j \to \int \varphi \, d\mu.$$

Here $C_0(\mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R}^n$ with compact support. The following weak compactness theorem is very important, though not very deep. It follows rather easily from the separability of the space $C_0(\mathbb{R}^n)$.

**Theorem 9.2.** Suppose $\mu_j, j = 1, 2 \dots,$ are Borel measures on $\mathbb{R}^n$ such that for every compact set $K \subset \mathbb{R}^n, \sup_j \mu_j(K) < \infty$. Then the sequence $(\mu_j)$ has a weakly converging subsequence.

Examples:

*Example* 9.3. For Dirac measures $\delta_{a_j}, a_j \in \mathbb{R}^n$:
$\delta_{a_j} \to \delta_a$ if $a_j \to a \in \mathbb{R}^n$,
$\delta_{a_j} \to 0$ if $|a_j| \to \infty$.

Recall the restriction measure $\mu \llcorner A$:

$$\mu \llcorner A(B) = \mu(A \cap B) \text{ for } B \subset \mathbb{R}^n.$$

It is a Borel measure when $\mu$ is a Borel measure and $A$ is $\mu$ measurable.

*Example* 9.4. $\frac{1}{k} \sum_{j=1}^{k} \delta_{j/k} \to \mathcal{L}^1 \llcorner [0, 1]$ when $k \to \infty$.

Some basic properties:
Suppose that $\mu_j, \mu$ are Borel measures on $\mathbb{R}^n$ and $\mu_j \to \mu$ weakly. Then
(1) $\mu_j(A) \to \mu(A)$ if $\mu(\partial A) = 0$ for bounded sets $A$. However, $\mu_j(A)$ need not converge to $\mu(A)$ even for compact sets in general.
(2) $\mu(U) \leq \liminf_{j \to \infty} \mu_j(U)$ for open sets $U$.
(3) $\limsup_{j \to \infty} \mu_j(K) \leq \mu(K)$ for compact sets $K$.

### Tangent measures

Tangent measures of a measure $\mu$ tell us how $\mu$ looks locally. To define them we use the affine maps $T_{a,r}$ sending the ball $B(a, r)$ to the unit ball $B(0, 1)$:

$$T_{a,r}(x) = \frac{x - a}{r} \text{ for } x \in \mathbb{R}^n.$$

We also need image, or push forward, of a measure $\mu$ under a map $f$:

$$f_{\#}\mu(B) = \mu(f^{-1}(B)) \text{ for } B \subset \mathbb{R}^n.$$

If $\mu$ is a Borel measure and $f$ is continuous, $f_{\#}\mu$ is a Borel measure. In that case an equivalent definition is

$$\int \varphi \, df_{\#}\mu = \int \varphi(x) \, d\mu x \text{ for all } \varphi \in C_0(\mathbb{R}^n).$$

**Definition 9.5.** Let $\mu$ and $\nu$ be locally finite Borel measures on $\mathbb{R}^n$, not identically zero. We say that $\nu$ is a *tangent measure* of $\mu$ at $a \in \mathbb{R}^n$ if there are $c_j > 0, r_j > 0$, such that $\lim_{j\to\infty} r_j = 0$ and

$$c_j T_{a,r_j\#}\mu \to \nu \text{ weakly as } j \to \infty.$$

We then denote $\nu \in Tan(\mu, a)$.

This is equivalent to

$$\int \varphi\left(\frac{x-a}{r}\right) d\mu x \to \int \varphi \, d\nu \text{ for all } \varphi \in C_0(\mathbb{R}^n).$$

Often, but not always, one can choose $c_j = c/\mu(B(a, r_j))$.
Examples:

*Example* 9.6. If $A \subset \mathbb{R}^n$ is Lebesgue measurable, then for $\mathcal{L}^n$ almost all $a \in \mathbb{R}^n$,

$$Tan(\mathcal{L}^n \llcorner A, a) = \{c\mathcal{L}^n : 0 < c < \infty\}.$$

This follows from the Lebesgue density theorem.

*Example* 9.7. If $\Gamma \subset \mathbb{R}^n$ is a rectifiable curve, then for $\mathcal{H}^1$ almost all $a \in \Gamma$,

$$Tan(\mathcal{H}^1 \llcorner \Gamma, a) = \{c\mathcal{H}^1 \llcorner L_a : 0 < c < \infty\},$$

where $L_a \in G(n, 1)$ is the tangent line of $\Gamma$ at $a$.

More generally, if $E \subset \mathbb{R}^n$ is $\mathcal{H}^m$ measurable and $m$ rectifiable with $\mathcal{H}^m(E) < \infty$, then for $\mathcal{H}^m$ almost all $a \in E$,

$$Tan(\mathcal{H}^m \llcorner E, a) = \{c\mathcal{H}^m \llcorner V_a : 0 < c < \infty\},$$

where $V_a \in G(n, m)$ is the approximate tangent $m$ plane of $E$ at $a$.

*Example* 9.8. Let $C(1/4)$ be the Cantor set as in Example 6.7. The tangent measures of $\mathcal{H}^1 \llcorner C(1/4)$ are all of the form $c\mathcal{H}^1 \llcorner C$ where $C$ is an unbounded Cantor set which locally looks like $C(1/4)$.

Example 9.6 is a special case of the following lemma. We define the *support* of a Borel measure $\mu$ as

$$\operatorname{spt} \mu = \{x \in \mathbb{R}^n : \mu(B(x, r)) > 0 \text{ for all } r > 0\}.$$

Then $\operatorname{spt} \mu$ is closed, $\mu(\mathbb{R}^n \setminus \operatorname{spt} \mu) = 0$, and it is the smallest closed set $F$ with $\mu(\mathbb{R}^n \setminus F) = 0$.

**Lemma 9.9.** Let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n, a \in \mathbb{R}^n$ and let $B$ be a $\mu$ measurable set such that

$$\lim_{r\to 0} \frac{\mu(B(a,r)\setminus B)}{\mu(B(a,r))} = 0.$$

Then $Tan(\mu \llcorner B, a) = Tan(\mu, a)$. In particular, this holds for $\mu$ almost all $a \in \mathbb{R}^n$.

The second statement follows from the fact that the density theorem is valid for locally finite Borel measures on $\mathbb{R}^n$.

**Lemma 9.10.** Let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n$, not identically zero. Then $Tan(\mu, a) \neq \varnothing$ for $\mu$ almost all $a \in \mathbb{R}^n$.

This is a rather easy consequence of the compactness theorem 9.2 if $\mu$ satisfies for $\mu$ almost all $a \in \mathbb{R}^n$ the asymptotic doubling condition:

$$\limsup_{r\to 0} \frac{\mu(B(a,2r))}{\mu(B(a,r))} < \infty. \tag{9.1}$$

Without this the proof is a bit trickier.

The doubling condition (9.1) implies that $0 \in \operatorname{spt}\nu$ for all $\nu \in Tan(\mu, a)$. In general this is not true.

**Definition 9.11.** The $s$-dimensional *upper density* of a Borel measure $\mu$ at $x \in \mathbb{R}^n$ is

$$\Theta^{*,s}(\mu, x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{\alpha(s)r^s},$$

the $s$-dimensional *lower density* of $\mu$ at $x$ is

$$\Theta^{s}_{*}(\mu, x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{\alpha(s)r^s},$$

and the $s$-dimensional *density* of $\mu$ at $x$ is

$$\Theta^{s}(\mu, x) = \lim_{r\to 0} \frac{\mu(B(x,r))}{\alpha(s)r^s},$$

if the limit exists.

When $\mu = \mathcal{H}^s \llcorner A$ this agrees with Definition 1.3 of the $s$-densities for sets. We shall be mainly interested in measures $\mu$ for which $0 < \Theta^s_*(\mu, a) \leq \Theta^{*,s}(\mu, a) < \infty$ for $\mu$ almost all $a \in \mathbb{R}^n$. They satisfy also the doubling condition (9.1) and hence have tangent measures almost everywhere. Moreover, in this case the normalization constants $c_j$ of Definition 9.5 can be taken as $cr_j^{-s}$:

**Definition 9.12.** Let $0 < s < \infty$ and let $\mu$ and $\nu$ be locally finite Borel measures on $\mathbb{R}^n$, not identically zero. We say that $\nu$ is an *s-tangent measure* of $\mu$ at $a \in \mathbb{R}^n$ if the there are $c > 0, r_j > 0$, such that $\lim_{j\to\infty} r_j = 0$ and

$$cr_j^{-s} T_{a,r_j\#}\mu \to \nu \text{ weakly as } j \to \infty.$$

We denote then $\nu \in Tan_s(\mu, a)$.

**Lemma 9.13.** Let $0 < s < \infty$ and let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n$, not identically zero. If $0 < \Theta^s_*(\mu, a) \leq \Theta^{*,s}(\mu, a) < \infty$ for $\mu$ almost all $a \in \mathbb{R}^n$, then

$$Tan(\mu, a) = Tan_s(\mu, a) \text{ for } \mu \text{ almost all } a \in \mathbb{R}^n.$$

**Definition 9.14.** Let $0 < s < \infty$ and let $\nu$ be a locally finite Borel measures on $\mathbb{R}^n$, not identically zero. We say that $\nu$ is *s-uniform* if there is $c > 0$ such that

$$\nu(B(x, r)) = cr^s \text{ for all } x \in \operatorname{spt} \nu, r > 0.$$

**Lemma 9.15.** Let $0 < s < \infty$ and let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n$, not identically zero. Let $A$ be the set of $a \in \mathbb{R}^n$ such that $0 < \Theta^s_*(\mu, a) \leq \Theta^{*,s}(\mu, a) < \infty$ and set $t(a) = \Theta^{*,s}(\mu, a)/\Theta^s_*(\mu, a)$ for $a \in A$. Then for $\mu$ almost all $a \in A$, for every $\nu \in Tan(\mu, a)$ there is $0 < c < \infty$ such that

$$ct(a)r^s \leq \nu(B(x, r)) \leq cr^s \text{ for } x \in \operatorname{spt} \nu, r > 0.$$

In particular, if the positive and finite density $\Theta^s(\mu, a)$ exists for $a \in A$, then for $\mu$ almost all $a \in A$ every tangent measure of $\mu$ at $a$ is $s$-uniform.

The main result of this section is the following *Marstrand's theorem*:

**Theorem 9.16.** Let $0 < s < \infty$ and suppose that there exists a locally finite Borel measure $\mu$ on $\mathbb{R}^n$, not identically zero, such that the positive and finite density $\Theta^s(\mu, a)$ exists for $\mu$ almost all $a \in \mathbb{R}^n$. Then $s$ must be an integer and $0 \leq s \leq n$.

That $0 \leq s \leq n$ is very easy and the main point is that the density fails to exist for non-integral dimensional measures. Due to Lemmas 9.11 and 9.15 it is enough to show that $s$-uniform measures can only exist when $s$ is an integer. The proof of this will be given during the lectures. It can also be found in [LY] and [Ma].

## 10. Tangent measures, densities and rectifiability

Tangent measures were introduced by David Preiss in 1987. His main motivation was that he used them to prove the following theorem:

**Theorem 10.1.** Let $0 < m < \infty$ be an integer and suppose that $\mu$ is a locally finite Borel measure on $\mathbb{R}^n$ such that the positive and finite density $\Theta^m(\mu, x)$ exists for $\mu$ almost all $x \in \mathbb{R}^n$. Then $\mu$ is $m$ rectifiable, that is, there is an $m$ rectifiable set $E$ such that $\mu(\mathbb{R}^n \setminus E) = 0$.

This theorem is equivalent to its set version where $\mu = \mathcal{H}^m \llcorner E$. But for sets one can say more:

**Theorem 10.2.** Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$. Then the following are equivalent:
(1) $E$ is $m$ rectifibale,
(2) $\Theta^m(E, x) = 1$ for $\mathcal{H}^m$ almost all $x \in E$,
(3) $0 < \Theta^m(E, x) < \infty$ for $\mathcal{H}^m$ almost all $x \in E$.

The last statement should be read as "the limit $\lim_{r\to 0} \frac{\mathcal{H}^m(E\cap B(x,r))}{\alpha(m)r^m}$ exists and is positive and finite for $\mathcal{H}^m$ almost all $x \in E$".

The proof of these theorems is very complicated. I shall only discuss some ideas in the lectures. A brief sketch is given in [Ma], Chapter 17. The book of Camillo De Lellis, Rectifiable sets, Densities and Tangent Measures, EMS, 2008, is perhaps more easily accessible than Preiss's original paper, but it is still tough.

Here are the main steps of the proof: First one can prove a tangent measure characterization of rectifiability.

**Theorem 10.3.** Let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n$ such that $\Theta^m_*(\mu, x) > 0$ for $\mu$ almost all $x \in \mathbb{R}^n$. Then the following statements are equivalent:
(1) $\mu$ is $m$ rectifiable,
(2) for $\mu$ almost all $a \in \mathbb{R}^n$, there is $V_a \in G(n, m)$ such that

$$Tan(\mu, a) = \{c\mathcal{H}^m \llcorner V_a : 0 < c < \infty\},$$

(3) for $\mu$ almost all $a \in \mathbb{R}^n$,

$$Tan(\mu, a) \subset \{c\mathcal{H}^m \llcorner V : V \in G(n, m), 0 < c < \infty\}.$$

Let us say that a measure $\nu$ is $m$-flat if $\nu = c\mathcal{H}^m \llcorner V$ for some $V \in G(n, m), 0 < c < \infty$.

The equivalence of (1) and (2) is essentially the same as Theorem 7.2. Notice the difference between (2) and (3): (2) says that around almost all points $\mu$ can be well approximated by one $m$-flat tangent measure at all small scales, whereas (3) only tells us that for all small scales there is some $m$-flat measure, depending on the scale, appoximating $\mu$ well.

Of course, (2) implies (1). The main content of the theorem is the rather difficult statement, essentially due to John Marstrand from the 1960's, that (3) implies (1).

Using Theorem 10.3 the proof of Theorem 10.1 is now reduced to showing that the tangent measures of $\mu$ at almost all points are $m$-flat. By Lemma 9.15 we know that they are $m$-uniform. So if we could prove that $m$-uniform measures are $m$-flat, we would be done. This is true when $m = 1$ and $m = 2$:

**Theorem 10.4.** When $m = 1$ or $m = 2$, every $m$-uniform Borel measure on $\mathbb{R}^n$ is $m$-flat.

The proof for $m = 1$ is not too hard, but it is rather difficult for $m = 2$. For $m \geq 3$, this theorem is false due to the following example of Preiss:
Consider the cone

$$C = \{(x_1, \dots, x_4) \in \mathbb{R}^4 : x_4^2 = x_1^2 + x_2^2 + x_3^2\}.$$

Then $\mathcal{H}^3 \llcorner C$ is 3-uniform. In fact, up to translation and rotation it is the only non-flat 3-uniform measure on $R^4$. For $n \geq 5$ and $m = n-1$, the same is true with $C$ replaced by $C \times \mathbb{R}^{n-4}$.

So when $m > 2$ one needs to do something more. There are three key ideas. First, and this is very difficult, all $m$-uniform measures are either $m$-flat or far away from $m$-flat measures, in a sense that can be described precisely in terms of certain metrics. Secondly, and this is not so difficult, $Tan(\mu, a)$ is connected, in a natural sense. Thirdly, again not so hard, at $\mu$ almost $a \in \mathbb{R}^n$, $Tan(\mu, a)$ contains some $m$-flat measures. Putting these three together yields that at $\mu$ almost $a \in \mathbb{R}^n$, $Tan(\mu, a)$ contains only $m$-flat measures, and completes the proof.

## 11. Functions of bounded variation and sets of finite perimeter

This topic is discussed in [EG] and also in
E. Giusti, Minimal surfaces and functions of bounded variation, and
W. P. Ziemer, Weakly differentiable functions.
Most of the theory below is due to E. de Giorgi from the 1950's.

Functions of bounded variation in one dimension are classical. In higher dimensions a distributional definition has turned out to be the best:

**Definition 11.1.** Let $U \subset \mathbb{R}^n$ be open. A function $f \in L^1(U)$ is of *bounded variation*,

$$f \in BV(U),$$

if

$$\|Df(U)\| := \sup\{\int_U f div\varphi : \varphi \in C_0^1(U; \mathbb{R}^n), |\varphi| \leq 1\} < \infty.$$

Here

$$div\varphi = \sum_{i=1}^n \partial_i \varphi_i, \varphi = (\varphi_1, \dots, \varphi_n),$$

$C_0^1(U; \mathbb{R}^n) = \{\varphi : U \to \mathbb{R}^n : \varphi$ is continuously differentiable and $\operatorname{spt}\varphi \subset U$ compact$\}$.

If $f \in C^1(U, \mathbb{R})$, then by partial integration for all $\varphi \in C_0^1(U; \mathbb{R}^n)$

$$\int_U f div\varphi = -\sum_{i=1}^n (\partial_i f)\varphi_i,$$

which iyelds that

$$\|Df(U)\| := \int_U |\nabla f|.$$

This is also true if $f$ belongs to the Sobolev space $W^{1,1}(U)$. Hence

$$W^{1,1}(U) \subset BV(U).$$

*Example* 11.2. Let $E \subset \mathbb{R}^n$ be a bounded set with smooth, for example $C^2$, boundary. Then in general the characteristic function $\chi_E \notin W^{1,1}(U)$, for example if $E$ is a ball.

Let $\varphi \in C_0^1(U;\mathbb{R}^n)$. Then by the Gauss-Green theorem

$$\int \chi_E div\varphi = \int_E div\varphi = \int_{\partial E} \varphi \cdot \nu_E d\mathcal{H}^{n-1} = \int_{U\cap\partial E} \varphi \cdot \nu_E d\mathcal{H}^{n-1},$$

where $\nu_E$ is the outward unit normal of $E$. Then if $|\varphi| \leq 1$,

$$|\int \chi_E div\varphi| \leq \mathcal{H}^{n-1}(U \cap \partial E) < \infty,$$

so

$$\|D\chi_E\|(U) \leq \mathcal{H}^{n-1}(U \cap \partial E) < \infty,$$

and $\chi_E \in BV(U)$. Moreover, we can choose $\varphi$ so that $\varphi = \nu_E$ on $U \cap \partial E$ except possibly for a set of very small $\mathcal{H}^{n-1}$ measure, which gives

$$\|D\chi_E\|(U) = \mathcal{H}^{n-1}(U \cap \partial E) < \infty.$$

**Definition 11.3.** A Lebesgue measurable set $E \subset \mathbb{R}^n$ has *finite perimeter* in $U$ if $\chi_E \in BV(U)$. $E$ has finite perimeter if $\chi_E \in BV(\mathbb{R}^n)$. The *perimeter* of $E$ in $U$ is

$$P(E,U) = \|D\chi_E\|(U) = \sup\{\int_E div\varphi : \varphi \in C_0^1(U;\mathbb{R}^n), |\varphi| \leq 1\},$$

and the perimeter of $E$ is

$$P(E) = \|D\chi_E\|(\mathbb{R}^n) = \sup\{\int_E div\varphi : \varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^n), |\varphi| \leq 1\}.$$

As we noted before $P(E) = \mathcal{H}^{n-1}(\partial E)$ for smooth bounded sets $E$. This is not true in general even if $\mathcal{H}^{n-1}(\partial E) < \infty$.

The Riesz representation theorem leads to the following theorem:

**Theorem 11.4.** Let $f \in BV(U)$. Then there are a finite Borel measure $\mu$ and a $\mu$ measurable function $\sigma : U \to \mathbb{R}^n$ such that

$$|\sigma(x)| = 1 \text{ for } \mu \text{ almost all } x \in \mathbb{R}^n,$$

$$\int_U f div\varphi = -\int_U \varphi \cdot \sigma \, d\mu \text{ for all } \varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^n).$$

**Notation:**

$$\|Df\| = \mu,\ Df = \sigma\|Df\|,$$

that is, $Df$ is a vector measure defined by

$$\int \varphi \cdot dDf = \int \varphi \cdot \sigma\, d\|Df\|,\ \varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^n).$$

So $Df$ is the distributional gradient of $f$ and its components $\sigma_i\|Df\|$ are the distributional partial derivatives of $f$. Thus $f \in BV(U)$ means that the the distributional partial derivatives of $f$ are finite Borel measures in $U$.

If $E \subset \mathbb{R}^n$ has *finite perimeter* in $U$, we set

$$\|\partial E\| = \|D\chi_E\|,\ \nu_E = -\sigma,$$

Hence

$$\int_E div\varphi = \int_U \varphi \cdot \nu_E\, d\|\partial E\|,\ \varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^n).$$

Thus in the disributional sense $\nu_E$ is the outer normal of $E$.

If $V \subset U$ is open with compact closure $\overline{V} \subset U$, then for $f \in BV(U)$ and $\chi_E \in BV(U)$,

$$\|Df\|(V) = \sup\{\int f div\varphi : \varphi \in C_0^1(V;\mathbb{R}^n), |\varphi| \le 1\}.$$

$$\|\partial E\|(V) = \sup\{\int_E div\varphi : \varphi \in C_0^1(V;\mathbb{R}^n), |\varphi| \le 1\}.$$

These determine uniquely the finite Borel measures $\|Df\|$ and $\|\partial E\|$.

The geometric measure part of this theory is to analyze $\|\partial E\|$ and $\nu_E$ geometrically. This was done by de Giorgi in the 1950's with applications to the minimal surfaces.

The following three theorems are basic theorems of the general theory. They are neither trivial nor very difficult.

**Theorem 11.5.** [Lower semicontinuity] Let $f_j \in BV(U)$ and $f_j \to f$ in $L^1(U)$. Then

$$\|Df\|(U) \le \liminf_{j\to\infty} \|Df_j\|(U).$$

**Theorem 11.6.** [Approximation] If $f \in BV(U)$, then there are $f_j \in C_0^\infty(U)$ such that

$$f_j \to f \text{ in } L^1(U),$$

$$\|Df_j\|(U) \to \|Df\|(U),$$

the measures $Df_j$ and $\|Df_j\|$ converge weakly to the measures $Df$ and $\|Df\|$.

**Theorem 11.7.** [Compactness] Let $f_j \in L^1(U)$ such that

$$\sup_j \left( \int_V |f_j| + \|Df_j\|(V) \right) < \infty$$

for every open $V \subset U$ with compact closure $\overline{V} \subset U$. Then there is a subsequence $(f_{j_i})$ and a locally integrable function $f$ in $U$ such that

$$f_{j_i} \to f \text{ in } L^1(V)$$

for every open $V \subset U$ with compact closure $\overline{V} \subset U$. Moreover, $f \in BV(V)$ for such $V$.

We proved earlier the coarea formula for Lipschitz functions. Here is one for BV-functions. Neither of them includes the other, why?

**Theorem 11.8.** [Coarea] Let $f \in BV(U)$ and for $t \in \mathbb{R}$,

$$E_t = \{x \in U : f(x) > t\}.$$

Then

(1) $E_t$ has finite perimeter for $\mathcal{L}^1$ almost all $t \in \mathbb{R}$,

(2)
$$\|Df\|(U) = \int_{-\infty}^{\infty} \|\partial E_t\|(U)\,dt.$$

(3) Conversely, if $f \in L^1(U)$ and
$$\int_{-\infty}^{\infty} \|\partial E_t\|(U)\,dt < \infty,$$
then $f \in BV(U)$.

**Sobolev, Poincaré and isoperimetric inequalities**

For smooth functions the following inequalities are classical. For BV-functions they follow with the aid of the approximation theorem.

**Theorem 11.9.** Let $f \in BV(\mathbb{R}^n), B \subset \mathbb{R}^n$ a ball and $\alpha = \mathcal{L}^n(B)^{-1}\int_B f$. Then

(1) $\left(\int |f|^{n/(n-1)}\right)^{(n-1)/n} \leq C(n)\|Df\|(\mathbb{R}^n)$, (Sobolev inequality)

(2) $\left(\int_B |f-\alpha|^{n/(n-1)}\right)^{(n-1)/n} \leq C(n)\|Df\|(B)$. (Poincaré inequality)

Applying these to $f = \chi_E$, we get the following isoperimetric inequalities:

**Theorem 11.10.** Let $E \subset \mathbb{R}^n$ be a bounded set with finite perimeter and $B \subset \mathbb{R}^n$ a ball. Then

(1) $\mathcal{L}^n(E)^{(n-1)/n} \leq C(n)P(E)$,

(2) $(\min\{\mathcal{L}^n(B \cap E), \mathcal{L}^n(B \setminus E)\})^{(n-1)/n} \leq C(n)\|\partial E\|(B)$.

**Reduced boundary**

The topological boundary of a set of finite perimeter may be all of $\mathbb{R}^n$, but the reduced boundary is more essential for applications.

**Definition 11.11.** Let $E \subset \mathbb{R}^n$ be of finite perimeter in $\mathbb{R}^n$. The *reduced boundary* $\partial^* E$ of $E$ consists of the points $x \in \mathbb{R}^n$ such that

(1) $\|\partial E\|(B)(B(x,r)) > 0\ \forall r > 0$ (that is, $x \in \operatorname{spt}\|\partial E\|$),

(2) $\lim_{r\to 0} \frac{\int_{B(x,r)} \nu_E\, d\|\partial E\|}{\|\partial E\|(B(x,r))} = \nu_E(x)$,

(3) $|\nu_E(x)| = 1$.

We then have rather easily

**Proposition 11.12.** Let $E \subset \mathbb{R}^n$ be of finite perimeter in $\mathbb{R}^n$. Then

(1) $\partial^* E \subset \partial E$,
(2) $\|\partial E\|(\mathbb{R}^n \setminus \partial^* E) = 0$,
(3) $\partial^*(\mathbb{R}^n \setminus E) = \partial^* E$.

**Theorem 11.13.** There are positive and finite numbers $c(n)$ and $C(n)$ such that the following holds. Let $E \subset \mathbb{R}^n$ be a set with finite perimeter. Then for all $x \in \partial^* E$,

(1) $\liminf_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap E)}{r^n} \geq c(n)$,
(2) $\liminf_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\setminus E)}{r^n} \geq c(n)$,
(3) $\liminf_{r\to 0} \frac{\|\partial E\|(B(x,r))}{r^{n-1}} \geq c(n)$,
(4) $\limsup_{r\to 0} \frac{\|\partial E\|(B(x,r))}{r^{n-1}} \leq C(n)$,
(4) $\limsup_{r\to 0} \frac{P(E\cap B(x,r))}{r^{n-1}} \leq C(n)$.

The following is the key lemma in the proof:

**Lemma 11.14.** Let $E \subset \mathbb{R}^n$ be a set with finite perimeter and $\varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$ and for $\mathcal{L}^1$ almost all $r > 0$,

$$\int_{B(x,r)\cap E} div\varphi = \int_{B(x,r)} \varphi\cdot\nu_E \, d\|\partial E\| + \int_{(\partial B(x,r))\cap E} \varphi\cdot\nu_{x,r}\, d\mathcal{H}^{n-1},$$

where $\nu_{x,r} = \frac{1}{r}(y-x)$ is the outward unit normal of $B(x,r)$.

As a corollary of Theorem 11.13 we obtain

**Corollary 11.15.** Let $E \subset \mathbb{R}^n$ be a set with finite perimeter. Then for all $A \subset \partial^* E$,

$$\frac{1}{C(n)}\mathcal{H}^{n-1}(A) \leq \|\partial E\|(A) \leq C(n)\mathcal{H}^{n-1}(A).$$

In particular, $\mathcal{H}^{n-1}(\partial^* E) < \infty$.

Next we want to show that $\partial^* E$ is $n-1$ rectifiable. To do this we show that it has approximate tangent plane almost everywhere. This follows from the following blow-up result which holds for all $x \in \partial^* E$:

**Theorem 11.16.** Let $E \subset \mathbb{R}^n$ be a set with finite perimeter and $x \in \partial^* E$. For $r > 0$, let

$$E_r = \{(z-x)/r : z \in E\} = \{y \in \mathbb{R}^n : ry + x \in E\},$$
$$H = \{y \in \mathbb{R}^n : \nu_E(x)\cdot y < 0\}.$$

Then $\chi_{E_r} \to \chi_H$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \to 0$, that is, for all $R > 0$,

$$\mathcal{L}^n\left(B(0,R)\cap[(E_r\setminus H)\cup(H\setminus E_r]\right) \to 0 \text{ as } r \to 0.$$

**Corollary 11.17.** For $x \in \partial^* E$,

$$\lim_{r\to 0}\frac{\mathcal{L}^n(B(x,r)\cap E)}{\alpha(n)r^n} = \lim_{r\to 0}\frac{\mathcal{L}^n(B(x,r)\setminus E)}{\alpha(n)r^n} = \frac{1}{2}.$$

**Theorem 11.18.** Let $E \subset \mathbb{R}^n$ be a set with finite perimeter. Then $\partial^* E$ is $n-1$ rectifiable and $\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial^* E$.

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