Geometric measure theory, Exercise 7, 6.11.2014

1. A function $f : \mathbb{R}^n \to \mathbb{R}$ has *approximate limit* $a \in \mathbb{R}$ at a point $x \in \mathbb{R}^n$, if for every $\varepsilon > 0$,

$$\lim_{r \to 0} \frac{\mathcal{L}^n(\{y \in B(x,r) : |f(y) - a| > \varepsilon\})}{\alpha(n) r^n} = 0.$$

As before $\alpha(n) = \mathcal{L}^n(B(0,1))$. Show that if f is Lebesgue measurable, this holds if an only if there is a Lebesgue measurable set $A \subset \mathbb{R}^n$ such that x is a density point of A, that is,

$$\lim_{r \to 0} \frac{\mathcal{L}^n(A \cap B(x,r))}{\alpha(n) r^n} = 1,$$

and

$$\lim_{y \to x, y \in A} f(y) = a.$$

2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *approximately continuous* at a point $x \in \mathbb{R}^n$, if it has approximate limit $f(x)$ at x. Show that any Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is approximately continuous almost everywhere.
Suggestion: use Lusin's theorem and the density point theorem.

It can also be shown that the converse holds: if f is approximately continuous almost everywhere, then it is measurable. Think about this.

3. Find infinitely many explicit (perhaps described from a picture) lines $L_1, L_2, \dots \in G(2,1)$ for which $\mathcal{H}^1(P_{L_j}(C(1/4))) = 0$, where $C(1/4)$ is the Cantor set of Example 6.7.

4. Let $\Gamma \subset \mathbb{R}^2$ be a rectifiable curve and $A \subset \Gamma$ with $\mathcal{H}^1(A) > 0$. Prove that there can exist at most one line $L \in G(2,1)$ such that $\mathcal{H}^1(P_L(A)) = 0$.
Suggestion: use the fact that Γ has an ordinary tangent line at almost all of its points and consider subsets of A where these tangent lines are very close to each other.

5. Let $A \subset \mathbb{R}^2$ be $\mathcal{H}^1$ measurable with $\mathcal{H}^1(A) < \infty$. Prove that A is purely 1 unrectifiable if and only if there are two different lines $L_1, L_2 \in G(2,1)$ for which $\mathcal{H}^1(P_{L_1}(A)) = \mathcal{H}^1(P_{L_2}(A)) = 0$.