Geometric measure theory, Exercise 6, 30.10.2014

1. Prove that if $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$ with $\Theta^{*,1}(A, a) > 0$, then A can have at most one 1 dimensional approximate tangent line at a.

2. Construct a set $A \subset \mathbb{R}^3$ with $\Theta^{*,2}(A, a) > 0$ such that every 2 dimensional plane containing the x_1-axis is an approximate tangent 2 plane for A at the origin.

3. Suppose that $A \subset \mathbb{R}^n$ and for some positive number $c, \mathcal{H}^m(A \cap B(a, r)) > cr^m$ for $a \in A, 0 < r < 1$. Show that every approximate tangent m plane for A is an ordinary tangent m plane for A.

4. If $a \in \mathbb{R}^2$ and $S \subset S^1$ is an arc on the unit circle, define the one-sided sector

$$Y(a, S) = \{a + te : t > 0, e \in S\}.$$

Prove the following analogue of Lemma 7.4 for one-sided sectors: Suppose $E \subset \mathbb{R}^2$ and $S \subset S^1$ is an arc. If

$$E \cap Y(a, S) = \varnothing \text{ for all } a \in E,$$

then E is 1 rectifiable.

5. Let $C \subset \mathbb{R}^2$ be compact, $\varepsilon > 0$ and $E = \{x \in \mathbb{R}^2 : d(x, C) = \varepsilon\}$. Prove that E is 1 rectifiable.
Hint: The previous exercise might help.