Geometric measure theory, Exercise 5, 16.10.2014

1. Prove that any countable union of m rectifiable sets is m rectifiable.

2. Prove that if $E \subset \mathbb{R}^n$ and for every $\varepsilon > 0$, there is an m rectifiable set $F \subset E$ such that $\mathcal{H}^m(E \setminus F) < \varepsilon$, then E is m rectifiable.

3. Prove that if $E \subset \mathbb{R}^n$ and E is m rectifiable, then there is an m rectifiable Borel set B such that $E \subset B$ and $\mathcal{H}^m(B) = \mathcal{H}^m(E)$.

The line $L \subset \mathbb{R}^2$ is called a tangent line for a set $A \subset \mathbb{R}^2$ at the point $a \in \mathbb{R}^2$ if for every $\alpha > 0$ there is $r > 0$ such that

$$A \cap B(a, r) \subset S(a, L, \alpha),$$

where $S(a, L, \alpha)$ is the two-sided sector

$$S(a, L, \alpha) = \{x \in \mathbb{R}^2 : d(x, L) \leq \alpha |x - a|\}.$$

4. Prove that if $\alpha : [a, b] \to \mathbb{R}^2$ is a continuously differentiable injective path, then the curve $\alpha([a, b])$ has a tangent line at all of its points.

5. Prove that the Cantor set $C(1/4)$ of Example 6.7 of the lecture notes does not have a tangent line at any of its points.