Geometric measure theory, Exercise 4, 9.10.2014

1. Prove that for any function $f : \mathbb{R} \to \mathbb{R}^n$, $\mathcal{H}^1(f(E)) = 0$, where

$$E = \{x \in \mathbb{R} : f \text{ is differentiable at } x \text{ and } f'(x) = 0\}.$$

Suggestion: look first at $f([-m, m] \cap E), m = 1, 2, \dots$.

2. What formula do you get when you apply the coarea formula with $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(x) = \max\{|x_1|, \dots, |x_n|\}?$$

We mean by *path* a continuous mapping $\alpha : [a, b] \to \mathbb{R}^n, a < b$. The *length* (or total variation) of α is

$$L(\alpha) := V_\alpha(a, b) := \sup\{\sum_{j=1}^{k} |\alpha(x_j) - \alpha(x_{j-1})| : a = x_0 < x_1 < \dots < x_k = b\}.$$

3. Prove that if $\alpha : [a, b] \to \mathbb{R}^n$ is injective, then $L(\alpha) \leq \mathcal{H}^1(\alpha([a, b]))$.

The path α is *rectifiable* if $L(\alpha) < \infty$.

4. Prove that if $\alpha : [a, b] \to \mathbb{R}^n$ is an injective rectifiable path and we set $V_\alpha(x) = V_\alpha(a, x)$ for $a \leq x \leq b$, then $V_\alpha : [a, b] \to [0, L(\alpha)]$ is a homeomorphism, $\beta = \alpha \circ V_\alpha^{-1} : [0, L(\alpha)] \to \mathbb{R}^n$ is Lipschitz with Lip$(\beta) \leq 1$ and $\beta([0, L(\alpha)]) = \alpha([a, b])$.

5. Prove further with the assumptions of Excecise 4 that in fact $|\beta'(x)| = 1$ for almost all $x \in [0, L(\alpha)]$, Lip$(\beta) = 1$ and $L(\alpha) = \mathcal{H}^1(\alpha([a, b]))$.