Geometric measure theory, Exercise 3, 2.10.2014

1. Prove that if $f : A \to \mathbb{R}, A \subset \mathbb{R}^n$, is Lipschitz, then $g : \mathbb{R}^n \to \mathbb{R}$,

$$g(x) = \inf\{f(y) + Lip(f)|x - y| : y \in A\}, x \in \mathbb{R}^n,$$

is Lipschitz with $g(x) = f(x)$ for $x \in A$ and $Lip(g) = Lip(f)$.

2. Prove that if $f : A \to \mathbb{R}^m$ is Lipschitz and $A \subset \mathbb{R}^n$ is Lebesgue measurable, then $f(A)$ is $\mathcal{H}^n$ measurable.
Hint: Prove this first for closed sets and appoximate A with closed subsets.

3. Prove that $\mathcal{H}^0$ is the counting measure: $\mathcal{H}^0(A)$ equals the number of points in A.

4. Prove that if $s \geq 0, f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $K \subset \mathbb{R}^n$ is compact, then the function $y \mapsto \mathcal{H}^s(K \cap f^{-1}\{y\}), y \in \mathbb{R}^m$, is a Borel function.
Hint: Prove this first for $\mathcal{H}^s_\delta$ in place of $\mathcal{H}^s$.

5. Prove that if $s \geq 0, f : A \to \mathbb{R}^m$ is Lipschitz and $A \subset \mathbb{R}^n$ is Lebesgue measurable, then $y \mapsto \mathcal{H}^s(A \cap f^{-1}\{y\}), y \in \mathbb{R}^m$, is $\mathcal{H}^n$ measurable.