Geometric measure theory, Exercise 2, 25.9.2014

1. Prove that if $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, is Lipschitz and $s \geq 0$, then

$$\mathcal{H}^s(f(A)) \leq Lip(f)^s \mathcal{H}^s(A).$$

What is the corresponding inequality for Hölder continuous maps f: for some $0 < \alpha < 1$ and $L < \infty$,

$$|f(x) - f(y)| \leq L|x-y|^\alpha.$$

2. Show that if $F \subset \mathbb{R}^n$ is closed, then the function $x \mapsto dist(x, F), x \in \mathbb{R}^n$, is Lipschitz.

3. Construct a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ which is not differentiable at any point of the $1/3$ Cantor set.

4. Let $A \subset \mathbb{R}$ be Lebesgue measurable with $\mathcal{L}^1(A) > 0$. Show that there exists a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that $f(A)$ is an interval.

5. Show that for the $1/3$ Cantor set $C \subset [0,1]$, $C + C = [0,2]$.