

Homework exercises set 7

1. Prove $ad_g([X, Y]) = [ad_g(X), ad_g(Y)]$ on any Lie group.
2. Locally, near the neutral element in a Lie group, one can write $\exp(X) \cdot \exp(Y) = \exp(Z)$. Show that in the case of a matrix group one has $Z = X + Y + \frac{1}{2}[X, Y] +$ higher order terms.
3. Let G be the group of real $n \times n$ matrices with zeros under the diagonal and one's on the diagonal. Show that the exponential function $\exp : Lie(G) \rightarrow G$ is a diffeomorphism.
4. In the case of the group $SU(2)$ one can use the parametrization $g = g(x) = e^{i\sigma \cdot x}$ for the group elements. Close to the unit element this is local chart on $SU(2)$. Here σ_i with $i = 1, 2, 3$ is a basis of traceless hermitean matrices (the Pauli matrices) and $x \in \mathbb{R}^3$. Derive an explicit formula for the adjoint action $\sigma \cdot y \mapsto g(x)\sigma \cdot yg(x)^{-1}$.
5. Show that the group $SL(2, \mathbb{R})$ of real 2×2 matrices with $\det = 1$ is diffeomorphic to the space $S^1 \times \mathbb{R}^2$. Show that the exponential map is not onto. Show that the Lie algebra of $SL(2, \mathbb{R})$ is isomorphic to the Lie algebra of $SO(2, 1)$. Here $SO(2, 1)$ is the group of linear transformations in \mathbb{R}^3 which preserve the pseudonorm $|x|^2 = x_1^2 + x_2^2 - x_3^2$ and have determinant = 1.