Homework exercises set 6

1. Compute the curvature tensor of the unit sphere S^2 from the Riemann metric induced by the euclidean metric in $\mathbb{R}^3$. Hint: The only independent components (in spherical coordinates) are $R^{\theta}_{\phi\theta\phi}$ and $R^{\phi}_{\theta\theta\phi}$ since R is antisymmetric in the last two of the lower indices.

2. Prove the relation $\Gamma^i_{ij} = \frac{1}{2}\partial_j \log \det(g)$ for the Levi-Civita connection.

3. It is sometimes useful to compute connection and curvature in a local basis different form a coordinate basis. That is, we can define Christoffel symbols

$$\nabla_a e_b = \Gamma^c_{ab} e_c,$$

where $e_1, e_2, \ldots, e_n$ is an arbitrary local basis of vector fields and $\nabla_a = \nabla_{e_a}$. Relate these Christoffel symbols to the coordinate basis Christoffel symbols Γ^k_{ij}. Let next e_1, e_2 be a local orthonormal basis of vector fields on S^2. Compute the curvature tensor in this basis. Compute also the curvature tensor on a sphere of radius r.

4. Show directly from the definition of parallel transport that the length of a parallel vector field along a curve is constant, in the case of Levi-Civita connection.

5. An a manifold M we have a global basis of orthonormal vector fields L_1, L_2, L_3 with nonzero commutation relations $[L_1, L_2] = L_3, [L_2, L_3] = L_1$, and $[L_3, L_1] = L_2$. Compute the Christoffel symbols for the Levi-Civita connection. Hint: Levi-Civita connection is the unique metric compatible torsion free connection on M. Use repeatedly the symmetry properties of Christoffel symbols.