Homework exercises set 5

1. Show that on the 2-dimensional torus $T^2 = S^1 \times S^1$ one can choose a complete system of coordinates such that the Christoffel symbols vanish identically on each coordinate patch.

2. Show that $f^*(d\omega) = d(f^*\omega)$ for any form ω on a manifold N and a smooth map $f : M \to N$.

3. We define Christoffel symbols on the unit sphere S^2 in terms of spherical coordinates (θ, ϕ), away from the poles $\theta = 0, \pi$, as

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2} \sin 2\theta \text{ and } \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot \theta$$

and all the other symbols equal to zero. Show that there is a globally defined covariant differentiation ∇ corresponding to these Christoffel symbols.

4. Show that the equations

$$\frac{dY^i(s)}{ds} + \Gamma^i_{jk}(x(s)) \frac{dx^j(s)}{ds} Y^k(x(s)) = 0$$

for parallel transport are consistent with coordinate transformations.

5. Compute directly from the definition the transformation rule for components of the torsion tensor in local coordinate transformations.