Homework exercises set 4

1. Let M be a pseudo-Riemannian manifold of signature (p, q), that is, the metric has p positive and q negative eigenvalues. Let $\omega \in \Omega^k(M)$. Show that $**\omega = \pm\omega$ and compute the phase $\pm$ as a function of p, q, and k.

2. Show that the definition of the integral of $\omega \in \Omega^n(M)$ over a n-dimensional manifold M does not depend on the choice of a partition of unity.

3. Let $\{\omega_{\alpha\beta}\}$ be a system of closed differential forms on a manifold M such that the support of each $\omega_{\alpha\beta}$ is in $U_\alpha \cap U_\beta$, where the sets U_α form a finite open cover of M. Assuming that

$$\omega_{\alpha\beta} + \omega_{\beta\gamma} = \omega_{\alpha\gamma}$$

on the triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ and $\omega_{\alpha\beta} = -\omega_{\beta\gamma}$, use a partition of unity to construct a closed differential form ω and potentials θ_α on U_α, $d\theta_\alpha = \omega$, such that on the overlaps $\theta_\alpha - \theta_\beta = \omega_{\alpha\beta}$.

4. Show the converse of the statement in Exercise 3: Given a closed form ω on M and an open cover such that ω is exact on each U_α construct the forms $\omega_{\alpha\beta}$.

5. Let U, V be a pair of open sets covering a compact manifold M. One can prove (easy) that there is an exact sequence of abelian groups and homomorphisms

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$$

where the second homomorphism is given by a restriction of differential forms, the third is given by the difference of the restrictions from U (resp. V) to $U \cap V$. This sequence induces a long exact sequence in cohomology,

$$\dots \to H^k(M) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to H^{k+1}(M) \to \dots$$

where the homomorphism to $H^{k+1}(M)$ is given by $\omega \mapsto -d(\rho_V \omega)$ on U and $\omega \mapsto d(\rho_U \omega)$ on V, using a partition of unity $1 = \rho_U + \rho_V$ subordinate to the covering U, V. Use this *Mayer -Vietoris sequence* to show that the de Rham cohomology $H^2(S^2) = \mathbb{R}$ for the unit sphere S^2.