Homework exercises, set 3

1. Show that $f_*[X,Y] = [f_*X, f_*Y]$.

2. Let X, Y be a pair of vector fields and ω, η differential forms on a manifold. Show that $i_{[X,Y]}\omega = \mathcal{L}_X(i_Y\omega) - i_Y(\mathcal{L}_X\omega)$ and that $i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^k \omega \wedge i_X\eta$ when ω is a k-form. Furthermore, observe that $i_X^2 = 0$ and $\mathcal{L}_X i_X = i_X \mathcal{L}_X$.

3. Let $X, Y \in \mathcal{D}^1(M)$ and $\omega \in \Omega(M)$. Show that $\mathcal{L}_X(\mathcal{L}_Y\omega) - \mathcal{L}_Y(\mathcal{L}_X\omega) = \mathcal{L}_{[X,Y]}\omega$. Hint: Do first the case $\omega \in \Omega^1(M)$ and then generalize to forms of arbitrary degree.

4. Prove the relation $\mathcal{L}_X = d \circ i_X + i_X \circ d$.

5. Use the result of exercise 4 to show that Lie derivative and exterior differentiation commute.

6. Let $M = \mathbb{R}^2 \setminus \{0\}$ and $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$. Show that ω is closed. Define $f(x,y) = \text{arc}\tan(y/x)$. Compute df. Is ω exact?