Homework exercises, set 2

1. Consider rotations around different coordinate axis in $\mathbb{R}^3$. They form 1-parameter groups acting on $\mathbb{R}^3$ and define flows for three different vector fields in $\mathbb{R}^3$. Write down the vector fields as first order linear differential operators.

2. In $\mathbb{R}^3$ we can parametrize 2-forms with a help of a (pseudo) vector. Using this parametrization write explicitly the wedge product of a 1-form and a 2-form.

3. Show that the wedge product $f \wedge g$ of $f \in \Omega^k(V)$ and $g \in \Omega^l(V)$ is really in $\Omega^{k+l}(V)$.

4. With the notation of (3), show that $f \wedge g = (-1)^{kl} g \wedge f$.

5. Prove that the wedge product is associative.

6. Prove the formula

$$\begin{aligned}(d\omega)(X_1, X_2, \ldots, X_{k+1}) = &\sum_i (-1)^i X_i \cdot \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})\end{aligned}$$

for the exterior derivative of a $k$ form, starting from the definition in terms of local coordinates. Here the hat means that the corresponding argument is deleted.