Homework exercises, set 10

1. Let $\rho : SU(n) \to \mathrm{Aut}(V)$ be a representation by real orthogonal matrices. Show that the form $\mathrm{tr}\,(g^{-1}dg)^{4k+1}$ on $SU(n)$ vanishes identically when the trace is computed in the representation space V.

2. Compute $\int \mathrm{tr}(g^{-1}dg)^3$ for the function $g : \mathbb{R}^3 \to SU(2)$ given by

$$g(x) = \exp(if(r)x_k\sigma_k/r)$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is any smooth function such that $f(0) = 0$ and $f(r)$ converges smoothly to the constant function π when $r \to \infty$. The σ_k's are the hermitean Pauli matrices with square $= 1$.

3. We can view the complex group $SL(2, \mathbb{C})$ as a principal G bundle over S^2, where G is the group of upper triangular matrices in $SL(2, \mathbb{C})$ and $\pi : SL(2, \mathbb{C}) \to SL(2, \mathbb{C})/G = S^2$ is the canonical projection. Construct a *holomorphic* transition function for this bundle on the extended equator in S^2 and show that the function can be reduced to a subgroup $\mathbb{C}^\times \subset G$ of diagonal matrices.

4. Construct a nontrivial $SU(2)$ bundle over $M = S^2 \times S^2$ and compute the integral of the second Chern class over M with respect to the defining representation of $SU(2)$ in $\mathbb{C}^2$. What is the minimal value of this integral for all nontrivial bundles?

5. As in the case of the unit sphere, show that the equivalence classes of principal $U(1)$ bundles over the torus $S^1 \times S^1$ are parametrized by a single integer n which is equal to $\frac{1}{2\pi i}\int F$, where F is the curvature form of a connection. Using a triangulation of a 2-dimensional connected manifold Σ, show that the integral of a curvature form over Σ of a $U(1)$ bundle is always $2\pi i$ times an integer.