Differential Equations II
Exercise 1, fall 2014

1. Solve by adapting the theory of linear equations of second order the following homogeneous equation of third order:

$$y^{(3)} + 4y'' + y' - 6y = 0.$$

A tip. Rational solutions to a polynomial equation by systematical try. It is not necessary to give reasons that all the solutions are obtained.

2. Let $y_1 : I \to \mathbf{R}$ and $y_2 :\to \mathbf{R}$ be differentiable functions in the interval I such that $W(y_1, y_2)(x) = 0$ and $y_1(x) \neq 0 \neq y_2(x)$ for all $x \in I$. Show that the functions y_1 and y_2 are then linearly dependent, that is, the equation $c_1y_1(x) + c_2y_2(x) = 0$, $x \in I$, is satisfied by some constants c_1, c_2, where at least one is $\neq 0$.

A tip. Form a differential equation (one) concerning the functions y_1 and y_2.

3. Is the function $f(x, y) = y^3 \cos x$ uniformly Lipschitz continuous with respect to y in the set $I \times J$, when

(a) $I = [0, 1]$ and $J = [0, 1]$,

(b) $I = \mathbf{R}$ and $J = [0, 1]$,

(c) $I = [0, 1]$ and $J = [0, \infty[$?

In a positive case give any valid Lipschitz constant; in a negative case just no is enough.

4. Determine in $\mathbf{R}^2$ maximal domains where the DE

$$y' = f(x, y) = x\sqrt[3]{y + 1}$$

satisfies the conditions of the local existence and uniqueness theorem (Theorem 4.4 in the lecture material). Give briefly reasons concerning the conditions, but maximality needs no consideration.

Remark. The function f is defined on the whole xy-plane $\mathbf{R}^2$.

5. (a) Give first four Picard's approximations to the IVP

$$y' = -y, \ y(0) = 2.$$

(b) Solve the IVP exactly and compare to the Picard's approximations. Observations?

6. (a) Give first four Picard's approximations to the IVP

$$y' = x^{5/3}y + 1,\ y(0) = 1.$$

(b) Solve the IVP also exactly (it remains one unsimplified integral).