Exercise 1 (chapter 1.4): Conditionally on $\Theta = \theta$, $\{Y_i\}_{i=1}^n$ are independent and identically distributed random variables that follow an exponential distribution with rate θ. The density of the exponential distribution is

$$p(y \mid \theta) = \theta \exp\{-\theta y\}\,, \qquad y > 0\,.$$

Let the prior on Θ be a Gamma distribution with shape $\alpha = 1$ and rate $\beta = 1$. There are two datasets:

1. $n = 5$ and $\bar{y} = n^{-1} \sum_{i=1}^n y_i = 0.25$
2. $n = 100$ and $\bar{y} = 0.25$

For both datasets, plot the prior, likelihood, the product of prior and likelihood as well as the posterior density (which happens to be a Gamma density).

Exercise 2 (chapter 1.4): For the statistical model from Exercise 1, find a closed form formula for the predictive density

$$p(y^* \mid y) = \int_\Theta p(y^*, \theta \mid y)\,\mathrm{d}\theta = \int_\Theta p(y^* \mid \theta) p(\theta \mid y)\,\mathrm{d}\theta$$

of a new observation y^*. Evaluate and plot the predictive density for the first dataset from Exercise 1 by setting up a grid for the y^* values.

Exercise 3 (chapter 2.7): The joint conditional distribution of Y^* and Θ factorizes as

$$p(y^*, \theta \mid y) = p(y^* \mid \theta) p(\theta \mid y)\,,$$

because the random variables Y and Y^* are conditionally independent given $\Theta = \theta$. Derive this results from the multiplication rule for conditional distributions.

Exercise 4 (chapter 2.10): Let the random variable X follow a Gamma distribution with shape $\alpha > 0$ and rate $\beta > 0$. There is only information about $Y = X^{-1}$. The distribution of Y is the Inverse-Gamma distribution with parameters α and β.

1. Find the density of Y using a change-of-variables:
$$f_Y(y) = f_X(x)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = f_X(g(y))|g'(y)| \text{ under the bijection } y = h(x) \Leftrightarrow x = g(y)$$
2. Find a formula for the mode (i.e. the maximum point) of the density of Y
3. Find the expectation $\mathbb{E}[Y]$ assuming $\alpha > 1$ using $\mathbb{E}\big[X^{-1}\big]$

Exercise 5 (chapter 2.10): Let the random variables $\{X_i\}_{i=1}^3$ follow independently Gamma distributions with shape $\alpha_1, \alpha_2, \alpha_3 > 0$ and rate $\beta_1 = \beta_2 = \beta_3 = 1$. Using a multivariate change-of-variables

$$Y_1 = \frac{X_1}{X_1 + X_2 + X_3} \qquad Y_2 = \frac{X_2}{X_1 + X_2 + X_3} \qquad S = X_1 + X_2 + X_3\,,$$

find the joint density of Y_1, Y_2 and S. Find also the joint density of Y_1 and Y_2 by integrating out S (which happens to be a Dirichlet distribution).

Exercise 6 (chapter 3.2): Let the random variable X follow a Pareto distribution with shape $\alpha > 0$ and scale $x_m > 0$. The density of the Pareto distribution is

$$f_X(x) = \frac{\alpha\, x_m^\alpha}{x^{\alpha+1}}\,, \qquad x \geq x_m\,.$$

Derive the inverse transformation method to simulate from the Pareto distribution (there is no function in the standard packages of R).

Exercise 7 (chapter 3.4): Let $f_X(x)$ be the density of a continuously distributed random variable X. The cumulative distribution $F_X(x)$ and quantile function $F_X^{-1}(u)$ with $u \in (0,1)$ are known. Derive the inverse transformation method when the distribution of X is truncated to the interval $I = (a, b)$ with $a < b$. The density of the truncated distribution is proportional to the unnormalized density

$$g_X^*(x) \propto f_X(x) 1_{(a,b)}(x)\,.$$

Start by determining the normalizing constant k such that $g_X(x) = g_X^*(x)/k$ is a density and then derive the cumulative distribution $G_X(x)$ and quantile function $G_X^{-1}(u)$ of the truncated distribution.