CONTINUOUS IMAGES AND THEIR PIXEL APPROXIMATIONS

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(Remark: the version of November 5 had errors in formula (1))

1. Continuous images

We let the image domain be the half-open unit square $\Omega = [0,1)^2 \subset \mathbb{R}^2$. In other words,

$$\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq x < 1,\ 0 \leq y < 1\}.$$

We model a monochrome (black&white) image with a (Lebesgue measurable) function $f : \Omega \to [0,1]$. Such an f is referred to as *continuous image* because it is defined on the infinite set Ω. The function f need not be continuous, though.

2. Pixel images

Square-shaped pixel images are defined as $n \times n$ matrices with non-negative real-valued elements smaller or equal to one. Throughout the text we take for simplicity $n = 2^m$ with $m > 0$.

For each resolution parameter $m = 1, 2, \ldots$ define pixels

$$\Omega^{(m)}_{k\ell} := \left\{ (x,y) \;\middle|\; \begin{array}{ccccc} (\ell-1)2^{-m} & \leq & x & < & \ell\, 2^{-m}, \\ 1 - k\, 2^{-m} & \leq & y & < & 1-(k-1)2^{-m} \end{array} \right\} \tag{1}$$

for $1 \leq k \leq 2^m$ and $1 \leq \ell \leq 2^m$. See Figure 1. The side of a pixel has length $D_m = 2^{-m}$, and the area of a pixel is

$$|\Omega^{(m)}_{k\ell}| = D_m^2 = 2^{-2m}.$$

Note that for fixed $m > 0$ the pixels are disjoint:

$$\Omega^{(m)}_{k\ell} \cap \Omega^{(m)}_{k',\ell'} = \emptyset \quad \text{if } k \neq k' \text{ or } \ell \neq \ell',$$

and they cover the image domain completely:

$$\Omega = \bigcup_{k=1}^{2^m} \bigcup_{\ell=1}^{2^m} \Omega^{(m)}_{k\ell}.$$

Also, any pixel $\Omega^{(m)}_{k\ell}$ is a union of exactly four pixels of the form $\Omega^{(m+1)}_{ij}$.

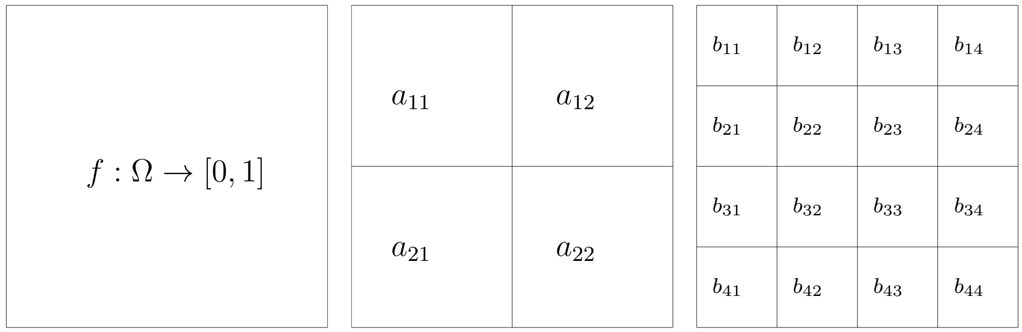

FIGURE 1. Left: image defined as a function on the half-open square $\Omega = [0,1)^2$. Middle and right: pixelizations for resolutions $n = 2^1$ and $n = 2^2$, respectively.

Given a continuous monochrome image f, the corresponding pixel image at resolution $m > 1$ is the $2^m \times 2^m$ matrix $A = [b_{k\ell}]$ with elements

$$b_{k\ell} := \frac{1}{|\Omega_{k\ell}^{(m)}|} \int_{\Omega_{k\ell}^{(m)}} f(x,y)\, dxdy = 2^{2m} \int_{\Omega_{k\ell}^{(m)}} f(x,y)\, dxdy. \tag{2}$$

3. DOWNSAMPLING

Given a continuous image f, we can consider pixelized versions of it at two different resolutions. For the sake of the argument, let $B = [b_{k\ell}]$ be a $2^m \times 2^m$ matrix and $A = [a_{ij}]$ be a $2^{m+1} \times 2^{m+1}$ matrix. Here both the elements $b_{k\ell}$ and the elements a_{ij} are defined using (2) at the appropriate resolutions.

Then it holds that

$$b_{k\ell} = \frac{1}{4}(a_{ij} + a_{i'j'} + a_{i''j''} + a_{i'''j'''}),$$

where the indices i, i', i'', i''' and j, j', j'', j''' satisfy

$$\Omega_{k\ell}^{(m)} = \Omega_{ij}^{(m+1)} \bigcup \Omega_{i'j'}^{(m+1)} \bigcup \Omega_{i''j''}^{(m+1)} \bigcup \Omega_{i'''j'''}^{(m+1)}).$$

For example,

$$b_{11} = \frac{1}{4}(a_{11} + a_{21} + a_{12} + a_{22}).$$