Introduction to mathematical physics: Quantum dynamics — Midterm Quiz, Thursday 17.10.2013

This quiz counts as one exercise sheet. You may use any results stated during the lectures and exercises to solve the problems.

Problem 1

Describe in a few sentences how the mathematical concept of Hilbert spaces is connected to physics of quantum systems. It was mentioned in the lecture notes that a spin-$\frac{1}{2}$ particle has a two-component wave function. Which of the following three choices can then be used as a Hilbert space for a spin-$\frac{1}{2}$ particle: $L^4(\mathbb{R}^3, \mathbb{C}^2)$, $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ or $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$? Why are the other two alternatives not relevant?

Problem 2

Every strongly continuous unitary semigroup has an *infinitesimal generator*. Below are six claims about their properties: for each claim, mark it as “true”, if you believe the claim to be true for every infinitesimal generator, and as “false”, otherwise.

(a) An infinitesimal generator is densely defined.
(b) An infinitesimal generator is a closed operator.
(c) An infinitesimal generator is a map from $[0, \infty)$ to a Hilbert space.
(d) An infinitesimal generator is a bounded operator.
(e) An infinitesimal generator has a spectral decomposition.
(f) An infinitesimal generator can be a function of time, the parameter of the semigroup.

Problem 3

Suppose T is a normal operator on a complex Hilbert space $\mathcal{H}$ and let E denote its spectral decomposition. As explained in the lecture notes, if $f : \mathbb{C} \to \mathbb{C}$ is Borel measurable, then we can define an operator $f(T)$ using E.

(a) If $\phi, \psi \in \mathcal{H}$, what is the connection between E and the scalar product $(\phi, f(T)\psi)$?
(b) What do you need to assume about f so that $f(T)$ is *normal*? How about *bounded*?
(c) What do you need to assume about T so that e^T is unitary? Justify your answer.

Problem 4

Explain how differential operators, such as ∂^2, can be defined as normal operators on $L^2(\mathbb{R})$. For which constants $c_0 \in \mathbb{C}$ is the operator $c_0\partial$ self-adjoint?

Problem 5

Do you have any comments or complaints about the course so far? What was the best part, what the worst? Do you have any wishes for the second half of the course? (For instance, is there any specific topic in quantum mechanics on whose mathematical theory you would be interested in?)