Introduction to mathematical physics: Quantum dynamics — Homework set 13, Monday 9.12.2013

This is the final Homework set. There will normal lectures on Thu 5.12. and Mon 9.12. and we will use the lecture on Thu 12.12. for an optional question session where you can ask about the projects and the course topics.

Exercise 1

Prove Proposition 12.2.5. about the properties of the antisymmetric and symmetric subspaces $\mathcal{H}_N^{\pm}$ of $\mathcal{H}_N := \bigotimes_{n=1}^{N} \mathfrak{h}$.

Exercise 2

Translation semigroups

Consider $N \geq 2$ spinless particles of mass $m_i > 0$, $i = 1, 2, \ldots, N$, and let $\mathcal{H} := L^2((\mathbb{R}^3)^N)$ denote the corresponding Hilbert space. A *translation* by $\mathbf{y} \in \mathbb{R}^3$ on $\mathcal{H}$ is defined via the formula

$$(\tau_{\mathbf{y}}\psi)(\mathbf{x}_1, \ldots, \mathbf{x}_N) := \psi(\mathbf{x}_1 - \mathbf{y}, \ldots, \mathbf{x}_N - \mathbf{y}), \quad x \in (\mathbb{R}^3)^N,\ \psi \in \mathcal{H}\,. \tag{1}$$

(a) Explain why (1) defines an operator on $\mathcal{H}$. Prove that every $\tau_{\mathbf{y}}$ is unitary.

(b) Show that $\tau_{\mathbf{y}}\tau_{\mathbf{y}'} = \tau_{\mathbf{y}+\mathbf{y}'}$ for all $\mathbf{y}, \mathbf{y}' \in \mathbb{R}^3$. When do the operators $\tau_{\mathbf{y}}$ and $\tau_{\mathbf{y}'}$ commute?

(c) Consider some fixed $\mathbf{y} \in \mathbb{R}^3$, and prove that $t \mapsto \tau_{t\mathbf{y}}$ defines a strongly continuous unitary semigroup.

Exercises 3

Translation invariant pair potentials

Consider N, m_i, $\mathcal{H}$ and $\tau_{\mathbf{y}}$, defined as in the previous Exercise. Assume that each pair (i', i), $i' \neq i$, of particles interacts via a potential which depends only their separation, as determined by the function $V_{i'i} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$.

(a) Prove that the resulting Hamiltonian

$$H := -\sum_{i=1}^{N} \frac{1}{2m_i}\nabla^2_{\mathbf{x}_i} + \sum_{i=1}^{N} \sum_{i'=1;i'\neq i}^{N} V_{i'i}(\mathbf{x}_{i'} - \mathbf{x}_i) \tag{2}$$

is self-adjoint on $\mathcal{H} := L^2((\mathbb{R}^3)^N)$ with $D(H) = D(H_0)$.

(b) Show that H is translation invariant: $\tau_{\mathbf{y}} H \subset H\tau_{\mathbf{y}}$, for every $\mathbf{y} \in \mathbb{R}^3$. (Hint: It suffices to check the translation invariance for test-functions. Explain why.)

(c) Suppose $\psi_0 \in \mathcal{H}$ is given and denote $\psi_t := \mathrm{e}^{-\mathrm{i}tH}\psi_0$ for $t > 0$. Consider some fixed $\mathbf{y} \in \mathbb{R}^3$, denote $\phi_0 := \tau_{\mathbf{y}}\psi_0$, and set then $\phi_t := \mathrm{e}^{-\mathrm{i}tH}\phi_0$ for $t > 0$. Show that $\phi_t = \tau_{\mathbf{y}}\psi_t$ for all $t \geq 0$. (Hint: spectral theory.)

(Please turn over)

Exercise 4

Let $\mathfrak{h}$ be a Hilbert space, and consider the standard Fock space generated by it: define $\mathcal{H}_0 = \mathbb{C}$, $\mathcal{H}_1 = \mathfrak{h}$, and $\mathcal{H}_N = \bigotimes_{n=1}^{N} \mathfrak{h}$, for $N = 2, 3, \ldots$, and then set $\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}_N$. Consider some fixed $g \in \mathfrak{h}$.

(a) For $N \in \mathbb{N}_+$ prove that there is a unique continuous linear map $a_N : \mathcal{H}_N \to \mathcal{H}_{N-1}$ with

$$a_N\left(\otimes_{n=1}^N \psi_n\right) = \sqrt{N}(g, \psi_1)_{\mathfrak{h}} \otimes_{n=2}^N \psi_n \,, \qquad \text{for all } \psi \in \mathfrak{h}^N := \textstyle\prod_{n=1}^N \mathfrak{h}\,.$$

(Hint: Theorem 2.12 and Exercise 2.4. Recall that for any non-zero $f \in \mathfrak{h}$ one can find an orthonormal basis of $\mathfrak{h}$ which contains $f/\|f\|$.)

(b) Show that $D_0 := \left\{\Psi \in \mathcal{F} \,\middle|\, \sum_{N=0}^\infty N\|\Psi_N\|^2 < \infty\right\}$ is a dense subspace of $\mathcal{F}$ which contains the vacuum vector $\Omega = (1, 0, 0, \ldots)$.

(c) Prove that the equation $(a\Psi)_N = a_{N+1}\Psi_{N+1}$, $N = 0, 1, \ldots$, defines an operator $D_0 \to \mathcal{F}$, and that this operator is *unbounded* if $g \neq 0$. Compute $a\Omega$.

(d) Show that there is a unique continuous linear map $c_N : \mathcal{H}_N \to \mathcal{H}_{N+1}$ with

$$c_N\left(\otimes_{n=1}^N \psi_n\right) = \sqrt{N+1}\, g \otimes \psi_1 \otimes \cdots \otimes \psi_N \,, \qquad \text{for all } \psi \in \mathfrak{h}^N\,,$$

for any choice of $N = 0, 1, \ldots$. Prove that if we set $(c\Psi)_0 = 0$ and $(c\Psi)_N = c_{N-1}\Psi_{N-1}$, for $N \in \mathbb{N}_+$, then c is an operator $D_0 \to \mathcal{F}$ which is unbounded if $g \neq 0$. Compute $c\Omega$.

$a = a(g)$ is called the *annihilation operator* related to g on $\mathcal{F}$ and $c = c(g)$ is called the *creation operator* related to g. Note that order is here important and it would be better to say that the operators annihilate and create a particle with the label “1”.

Exercise 5

Consider the fermionic Fock space defined in 12.2.6: $\mathcal{F}^{(-)} = \bigoplus_{N=0}^\infty \mathcal{H}_N^{(-)}$, where $\mathcal{H}_N^{(-)}$ is the totally antisymmetric subspace of $\mathcal{H}_N$. As before, let $P_N^{(-)}$ denote the orthogonal projection onto $\mathcal{H}_N^{(-)}$, and consider some fixed $g \in \mathfrak{h}$. The following statements show that the fermionic creation and annihilation operators, defined by restricting $a(g)$ and $c(g)$ to $\mathcal{F}^{(-)}$, are actually *bounded* operators.

(a) Show that the formulae $(P^{(-)}\Psi)_0 := \Psi_0$, $(P^{(-)}\Psi)_N := P_N^{(-)}\Psi_N$, for $N \in \mathbb{N}_+$, define an orthogonal projection $P^{(-)} : \mathcal{F} \to \mathcal{F}$ onto $\mathcal{F}^{(-)}$.

(b) Prove that $D_- := D_0 \cap \mathcal{F}^{(-)}$ is a dense subspace of $\mathcal{F}^{(-)}$, and consider the restrictions of $a(g)$ and $c(g)$ to $\mathcal{F}^{(-)}$, i.e., the maps $\tilde{a} := P^{(-)}a(g)|_{D_-}$ and $\tilde{c} := P^{(-)}c(g)|_{D_-}$. Show that there are unique $a_-(g), c_-(g) \in \mathcal{B}(\mathcal{F}^{(-)})$ such that $a_-(g)|_{D_-} = \tilde{a}$, $c_-(g)|_{D_-} = \tilde{c}$, and that then $\|a_-(g)\| = \|g\|_{\mathfrak{h}} = \|c_-(g)\|$. (Hint: What happens to $P_N^{(-)}(\otimes_{n=1}^N \psi_n)$, if $\psi_i = \psi_j$ for some $i \neq j$?)

(c) Show that $c_-(g)$ is the adjoint of $a_-(g)$. (In this context, usually denoted by $a_-^*(g)$.)