Introduction to mathematical physics: Quantum dynamics | Homework set 12, Monday 2.12.2013

Exercise 1

Choose a project topic for your "final exam" from the list given on the course webpage, and inform the lecturer about your choice by e-mail. If you have no particular preference, choose a few most promising topics, and I will make the selection for you. You can also suggest a topic outside the list.

Exercise 2

Consider some $d, d' \geq 1$ and denote $N := d+d'$. Suppose $\psi \in \mathcal{S}_N$ and assume $A \in \mathbb{R}^{N\times N}$ is an invertible matrix and $y \in \mathbb{R}^{d'}$. For every $z \in \mathbb{R}^d$ we can then identify $(z, y) \in \mathbb{R}^d \times \mathbb{R}^{d'} \cong \mathbb{R}^N$, and thus define

$$g(z) := \psi(A(z, y)) \,, \quad z \in \mathbb{R}^d.$$

Show that $g \in \mathcal{S}_d$. Is it also true in general if you do not assume A to be invertible? (This result was used in the proof of Theorem 11.4.1. Hint: Induction.)

Exercise 3

Suppose $N, M \in \mathbb{N}_+$ and assume that $\mathbf{R}_j \in \mathbb{R}^3$, $Z_j \in \mathbb{Z}$ are given for $j = 1, 2, \ldots, M$. Show that the operator $H := H_0 + \alpha V_{\mathrm{c}}$ is self-adjoint with $D(H) = D(H_0)$ for every $\alpha \in \mathbb{R}$ on the Hilbert space $L^2((\mathbb{R}^3)^N)$, and that it is essentially self-adjoint on the test-functions spaces $\mathcal{S}_{3N}$ and $\mathcal{D}_{3N}$.

Here V_{c} is defined as in the lecture notes: for $x = (\mathbf{x}_i)_{i=1}^N \in (\mathbb{R}^3)^N$ set

$$V_{\mathrm{c}}(x) := -\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{Z_j}{|\mathbf{x}_i - \mathbf{R}_j|} + \sum_{i=1}^{N}\sum_{i'=1}^{i-1} \frac{1}{|\mathbf{x}_{i'} - \mathbf{x}_i|} + \sum_{j=1}^{M}\sum_{j'=1}^{j-1} \frac{Z_{j'} Z_j}{|\mathbf{R}_{j'} - \mathbf{R}_j|} \,.$$

(This provides the mathematical definition for the "Molecular Hamiltonian" given in Section 11.4. Hint: Theorem 11.4.1.)

(Please turn over)

Exercise 4

The free Hamiltonian of a relativistic particle with a mass $m > 0$ (and using units in which the speed of light is one) is given by

$$H_{\mathrm{rel}} := \sqrt{-\nabla^2 + m^2}\,.$$

The operator is understood to be defined via the same construction as H_0 was in the lecture notes, i.e., using the corresponding multiplication operator in Fourier space. Hence, it defines a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^3)$.

The corresponding free Hamiltonian of a classical particle is given by $H_0 := -\frac{1}{2m}\nabla^2$. Show that $D(H_0) \subset D(H_{\mathrm{rel}})$, and that $\|(H_{\mathrm{rel}} - m)\psi\| \leq \|H_0\psi\|$ for all $\psi \in D(H_0)$. (Hint: $(a+b)(a-b) = a^2 - b^2$ for all $a, b \in \mathbb{C}$.)