Introduction to mathematical physics: Quantum dynamics

Homework set 11
Monday 25.11.2013

Exercise 1

Proof of Proposition 10.3

Let $d \le 3$, and consider $H_0 = -\frac{1}{2}\nabla^2$ on $L^2(\mathbb{R}^d)$. Prove that every $\psi \in D(H_0)$ can be chosen continuous. Show that for any $\varepsilon > 0$ there is $c_\varepsilon \ge 0$ such that for all $\psi \in D(H_0)$,

$$\sup_{x\in\mathbb{R}^d} |\psi(x)| = \|\psi\|_\infty \le \varepsilon \|H_0\psi\| + c_\varepsilon \|\psi\| \,. \tag{1}$$

(Hint: Recall the Riemann-Lebesgue lemma; in particular, note that by applying Remark "a)" on p. 73 of the lecture notes to the inverse Fourier transform one clearly has $\|\psi\|_\infty \le \|\mathcal{F}\psi\|_1$ for all $\psi \in L^2$. Show then that the assumptions imply $\mathcal{F}\psi \in L^1(\mathbb{R}^d)$, and try to prove the inequality (1) for some constants. After this, consider the functions f_r, with $r > 0$, defined via their Fourier transforms $(\mathcal{F}f_r)(k) := r^d(\mathcal{F}\psi)(rk)$, for $k \in \mathbb{R}^d$.)

Exercise 2

Let $d = 1$, and consider $H_0 = -\frac{1}{2}\nabla^2$ on $L^2(\mathbb{R})$. By Exercise 1 every element of $D(H_0)$ is then a continuous function. However, they are even more regular:

(a) Show that, if $\psi \in D(H_0)$, then $\psi \in C^{(1)}(\mathbb{R})$ and with $\phi := H_0\psi$ we have for all $x, x_0 \in \mathbb{R}$

$$\psi(x) = \psi(x_0) + (x - x_0)\psi'(x_0) - 2\int_{x_0}^{x} \mathrm{d}y\,(x-y)\phi(y)\,. \tag{2}$$

(b) Conversely, show that, if $\psi \in C^{(1)}(\mathbb{R})$ is such that $\psi, \psi' \in L^2(\mathbb{R})$ and there are some $x_0 \in \mathbb{R}$ and $\phi \in L^2(\mathbb{R})$ such that (2) holds for almost every $x \in \mathbb{R}$, then $\psi \in D(H_0)$ and $\phi = H_0\psi$.

(Hint: Recall that $-2H_0 = \partial_x\partial_x$ and use Exercise 10.3.)

Exercise 3

Let $d = 1$ and consider a real potential $V \in L^\infty(\mathbb{R})$ which is piecewise continuous: it is continuous apart from some isolated points in $Z \subset \mathbb{R}$. By Theorem 10.4, $H = H_0 + V$ is self-adjoint on $D(H_0)$.

(a) Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of H with an eigenvector ψ, that is, assume $\psi \in D(H)$ and $H\psi = \lambda\psi$. Show that then between any two consecutive points of discontinuity of V (that is, on any open interval in $\mathbb{R} \setminus Z$) ψ is twice continuously differentiable and satisfies the ordinary differential equation $-\frac{1}{2}\psi''(x) + V(x)\psi(x) = \lambda\psi(x)$. (Hint: Exercise 2.)

(b) State also the converse: Suppose $\lambda \in \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{C}$ satisfies $-\frac{1}{2}\psi''(x) + V(x)\psi(x) = \lambda\psi(x)$ on any open interval on which V is continuous (in particular, $\psi''(x)$ exists at every point of such intervals). What do you still need to check so that ψ is an eigenvector of H? Is it possible that the corresponding eigenvalue is not λ?

(Please turn over)

Exercise 4

Continuation from Exercise 3...

Consider H in the special case $V(x) = -E_0\mathbb{1}(|x| < a)$ where $a, E_0 > 0$ are some given parameters. (This setup is called a one-dimensional *potential well*.)

Assume λ is an eigenvalue of H, and find explicitly all of the corresponding eigenvectors. What is the dimension of the eigenspace? Derive also an implicit formula satisfied by any eigenvalue of H. Does H have eigenvalues for all $a, E_0 > 0$?

Exercise 5

Free Schrödinger operator on $(\mathbf{0}, \mathbf{1})$

In analogy to Exercise 10.5, define an operator T on $\mathcal{H} := L^2(\Omega)$ with $\Omega := (0,1)$ by using the domain

$$D(T) = \{\psi \in \mathcal{H} \,|\, \exists f \in D(H_0) \text{ such that } f|_\Omega = \psi\} \ ,$$

and setting $T\psi := H_0 f|_\Omega$ for $\psi \in D(T)$. Show that the definition makes sense: $T\psi$ does not depend on the choice of f.

For each of the domains D listed below, consider the restriction $A := T|_D$ of T on D. Show that every such A is a symmetric densely defined operator.

(a) **(Dirichlet boundary conditions)**

$$D = \{\psi \in D(T) \,|\, \psi(0) = 0,\ \psi(1) = 0\} \ .$$

(b) **(Mixed Dirichlet and Neumann)** There is $\alpha \in \mathbb{R}$ such that

$$D = \{\psi \in D(T) \,|\, \psi'(0) = \alpha\psi(0),\ \psi(1) = 0\} \,, \quad \text{or}$$
$$D = \{\psi \in D(T) \,|\, \psi(0) = 0,\ \psi'(1) = \alpha\psi(1)\} \ .$$

(c) **(General Neumann)** There are $\alpha, \beta \in \mathbb{R}$ such that

$$D = \{\psi \in D(T) \,|\, \psi'(0) = \alpha\psi(0),\ \psi'(1) = \beta\psi(1)\} \ .$$

(d) **(Generalized periodic)** There are $\varphi \in \mathbb{R}$ and a 2×2 matrix M, such that $M_{ij} \in \mathbb{R}$, $i, j = 1, 2$ and $\det M = 1$, and we define

$$D = \left\{\psi \in D(T) \middle| \begin{pmatrix} \psi(1) \\ \psi'(1) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\varphi} M \begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix} \right\} .$$

(All of these operators are actually self-adjoint. In fact, they yield the full collection of self-adjoint extensions in $L^2(\Omega)$ of the operator $f \mapsto -f''$ with the domain $C_c^\infty(\Omega)$. Therefore, they are all candidates for a possible definition of the generator of "free" evolution on $(0,1)$. Hint: try first to figure out why T is *not* self-adjoint.)